# Image Registration as an Optimization Problem

# Image Registration

**Overlaying two or more images of the same scene**

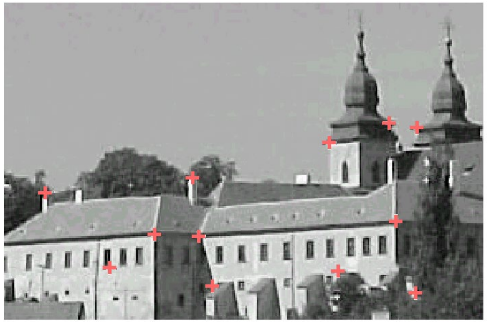
# Image Registration

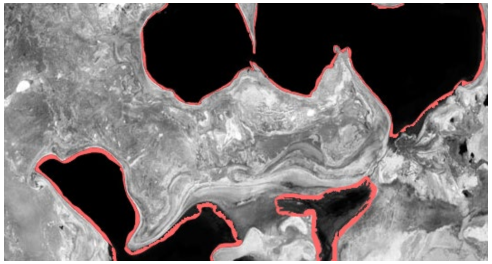
# IMAGE REGISTRATION METHODOLOGY

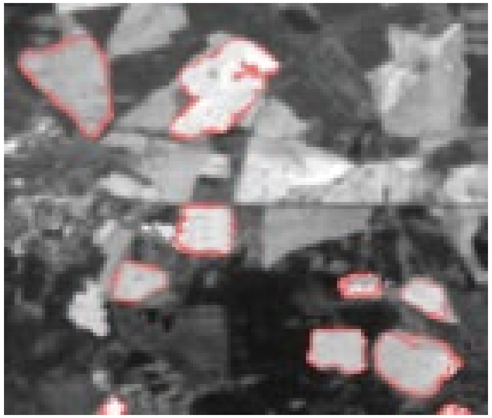
## Four basic steps of image registration

1. Control point selection
2. Control point matching
3. Transform model estimation
4. Image resampling
   and transformation

# CONTROL POINT SELECTION







- distinctive points
- corners
- lines
- closed-boundary regions
- virtual invariant regions
- window centers

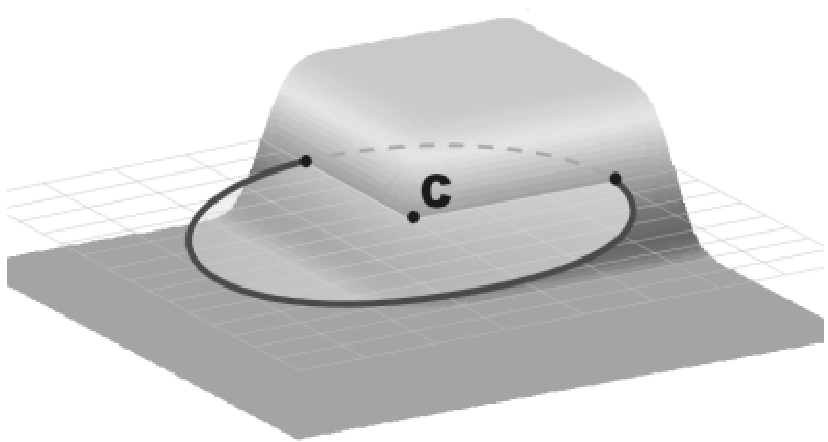
# CORNER DETECTION

Optimization problem:

Finding local extremes of a “cornerness” function

$$K(x,y) = \frac{f_x^2 f_{yy} - 2 f_x f_y f_{xy} + f_y^2 f_{xx}}{f_x^2 + f_y^2}$$

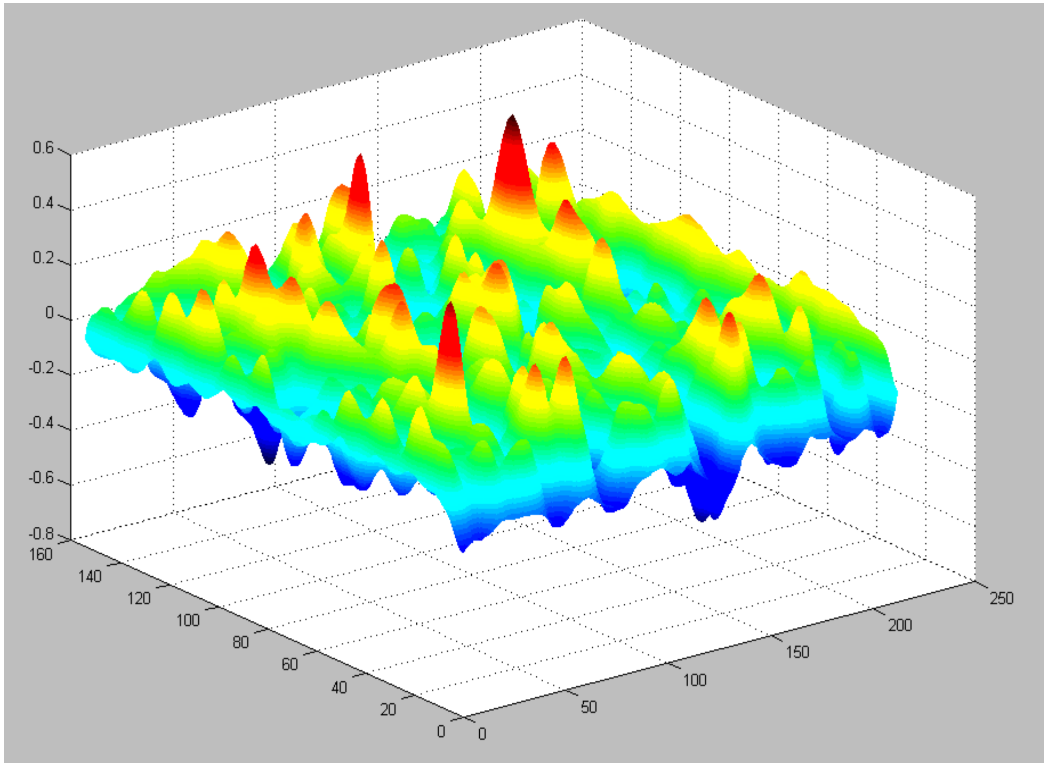
$$|H(x,y)| = f_{xx} f_{yy} - f_{xy}^2$$



# CORNER DETECTION

Optimization method: usually full search with constraints

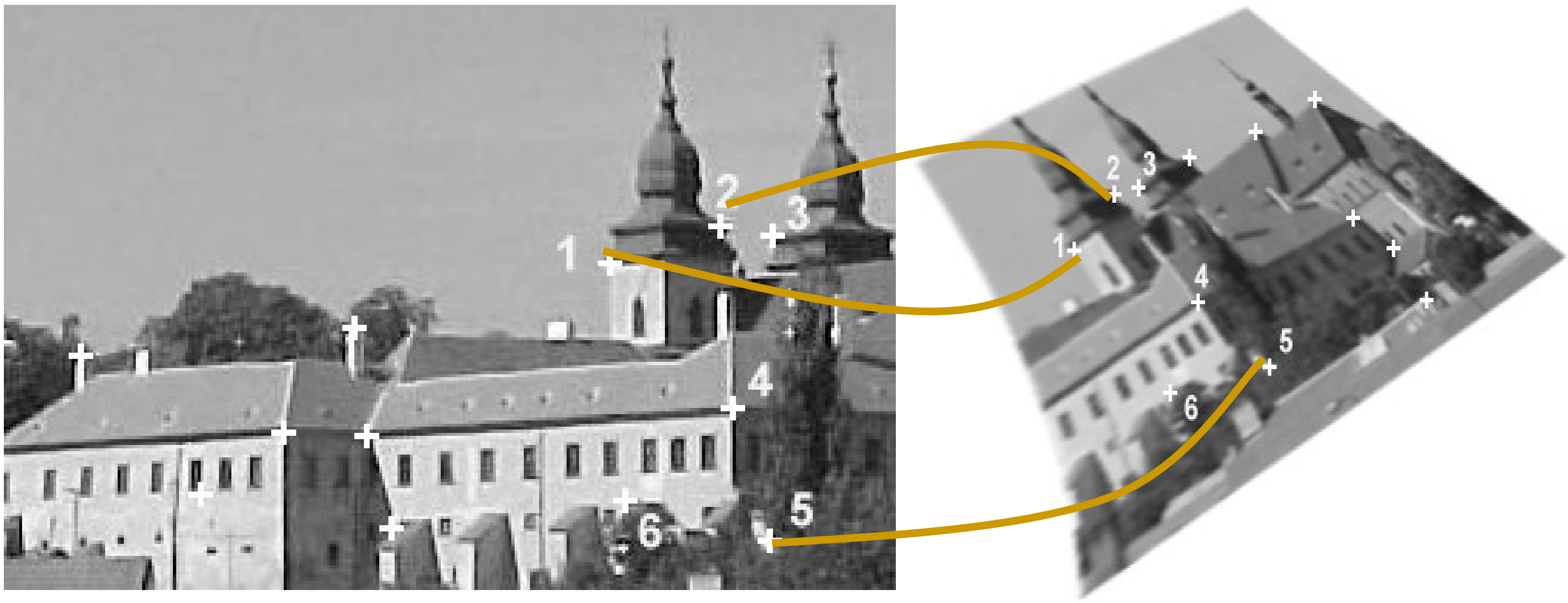
Cornerness functions use to have many extremes



# Corner detectors

Kitchen & Rosenfeld Harris Non-dif

# 2. Control point matching



# Control point matching

## Signal-based methods

Similarity measures calculated directly from the image graylevels

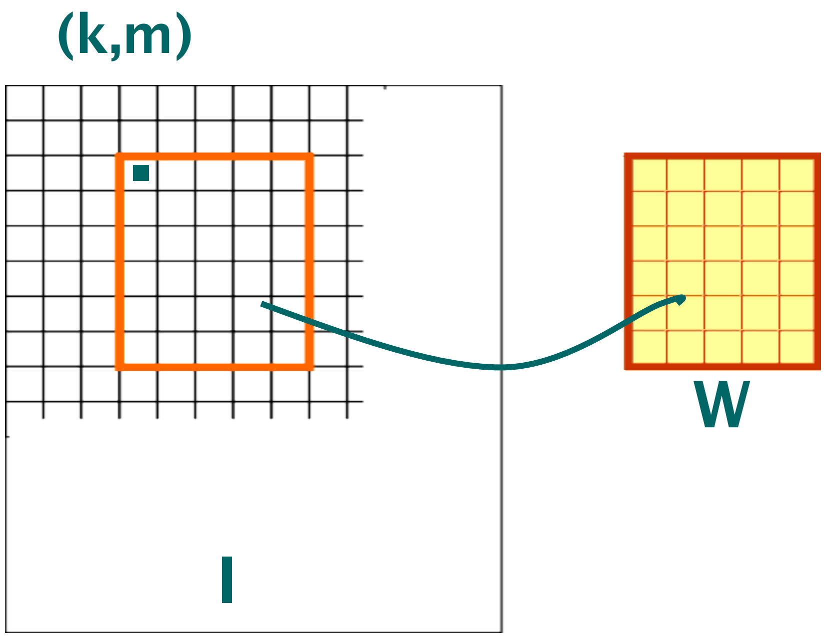
Examples - Image correlation, image differences, phase correlation, mutual information, …

## Feature-based methods

Symbolic description of the features

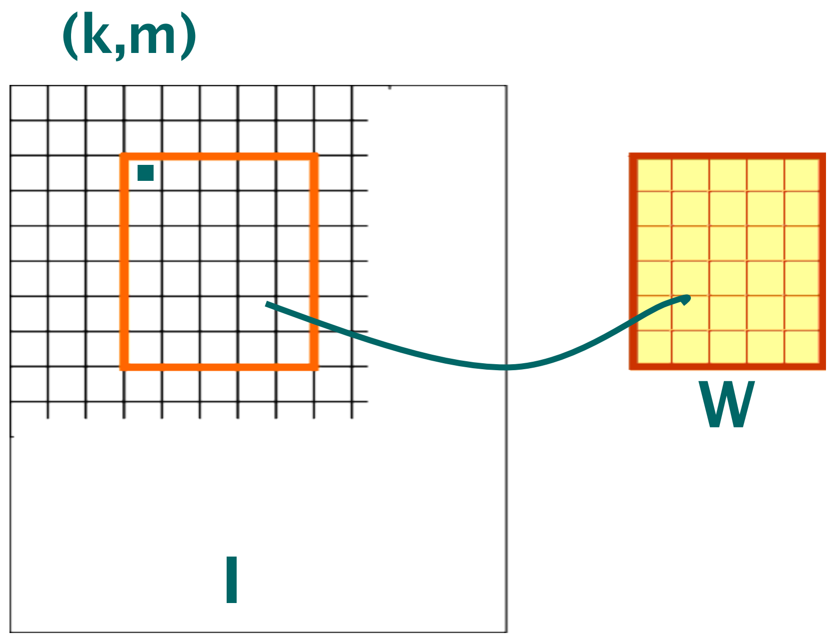
Matching in the feature space (classification)

# Image correlation




$$C(k,m) = \frac{\sum (I_{k,m} - \text{mean}(I_{k,m})) \cdot (W - \text{mean}(W))}{\sqrt{\sum (I_{k,m} - \text{mean}(I_{k,m}))^2} \cdot \sqrt{\sum (W - \text{mean}(W))^2}}$$

# Image correlation

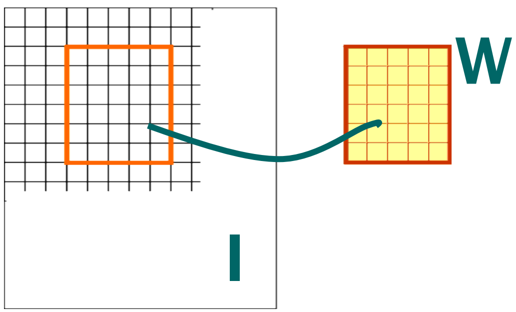



max C(k,m) – full search if W is small

- gradient-based methods for large W

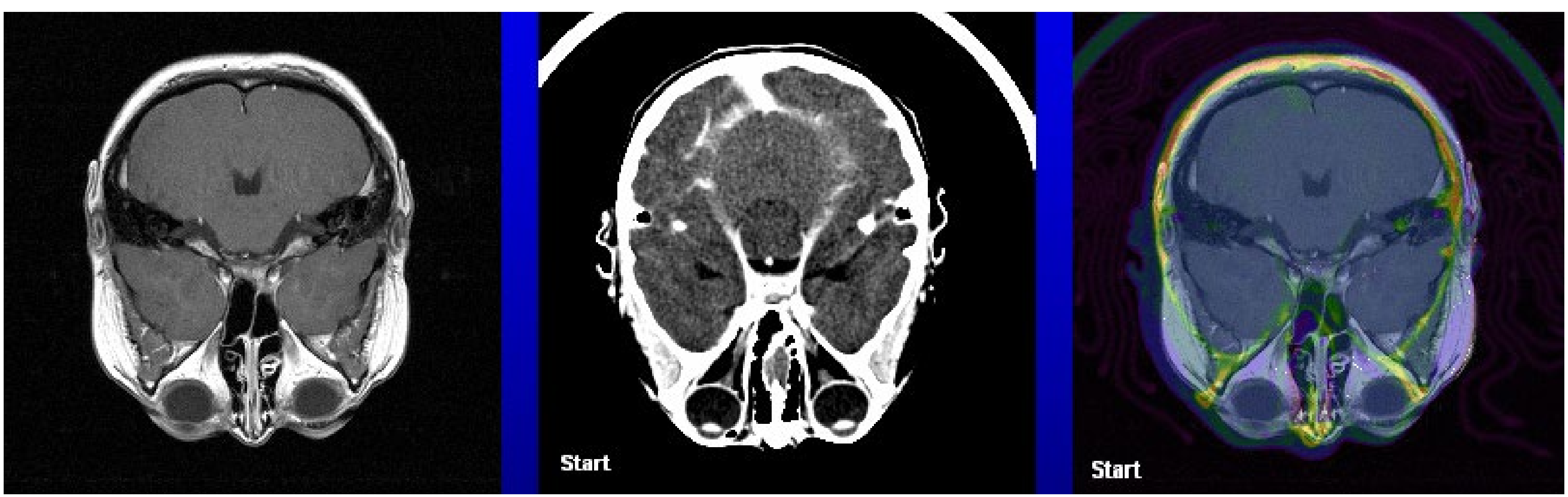
- the image must be spatially correlated

# Mutual information method




Statistical measure of the dependence between two images

$$MI(f,g) = H(f) + H(g) - H(f,g)$$



# MUTUAL INFORMATION

Entropy

$$H(X) = -\sum_{x} p(x) \log p(x)$$

Joint entropy

$$H(X,Y) = -\sum_{x} \sum_{y} p(x,y) \log p(x,y)$$

Mutual infomation

$$I(X;Y) = H(X) + H(Y) - H(X,Y)$$

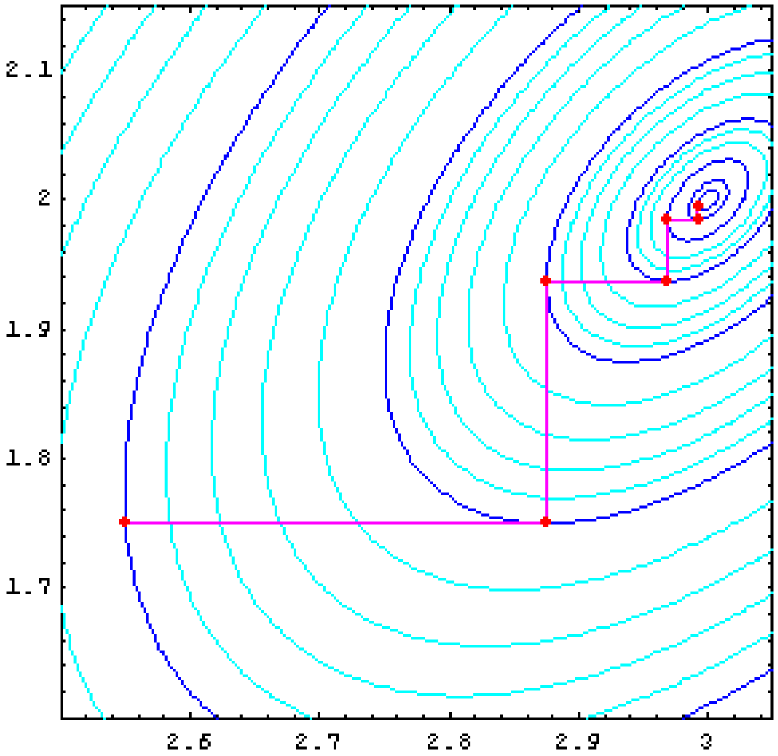
# MUTUAL INFORMATION

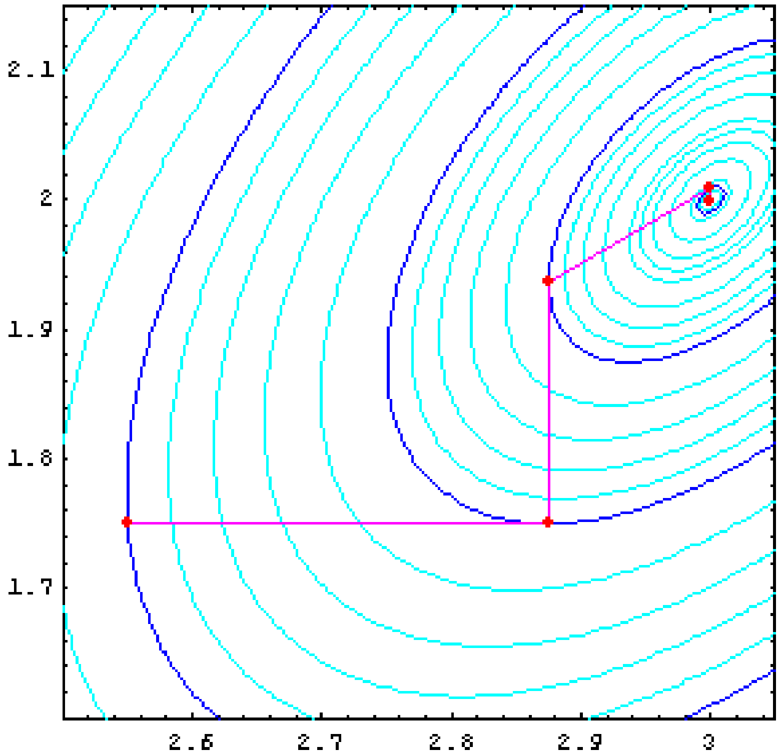
- Very often applied to 3D volume data
- Rotation estimation may be also required
- Good initial guess is available
- MI calculation is very time-consuming

- Full search is not feasible
- Sophisticated optimization algorithms

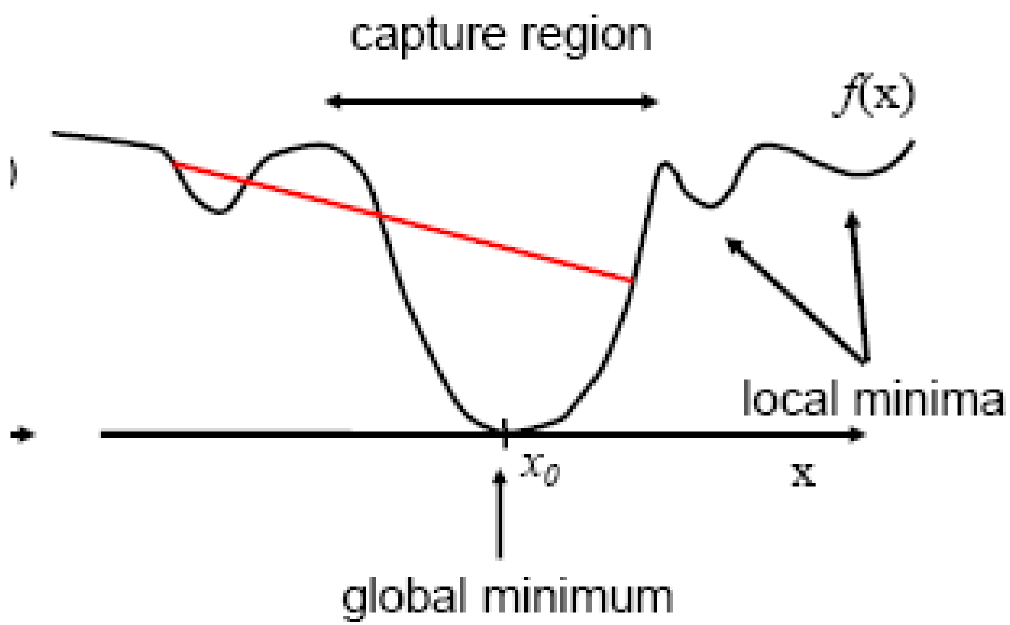
- Powell's optimization method

# Powell's optimization method



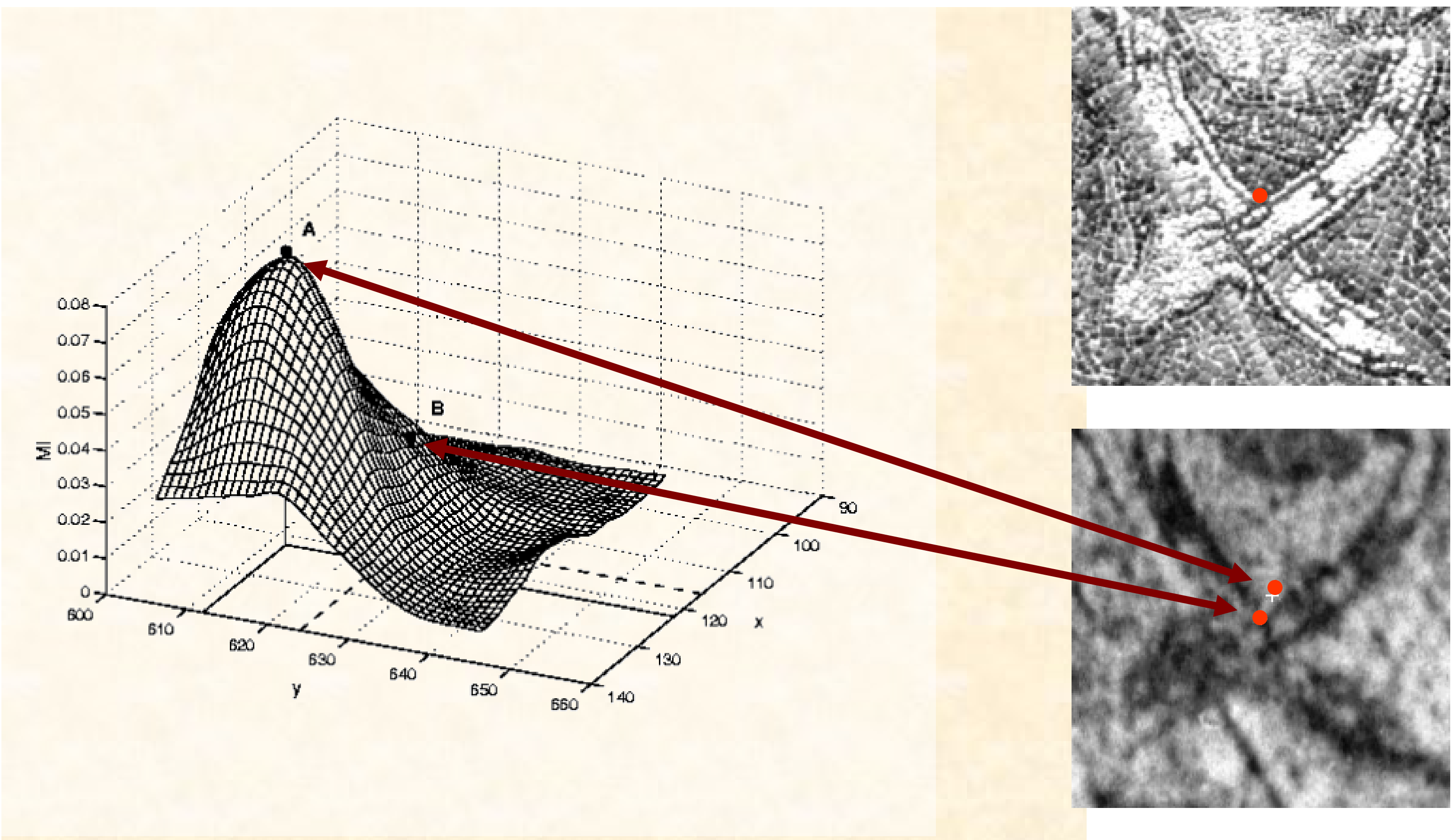




**The cab method** **Powell's method**


capture region
f(x)
local minima
x_0
x
global minimum

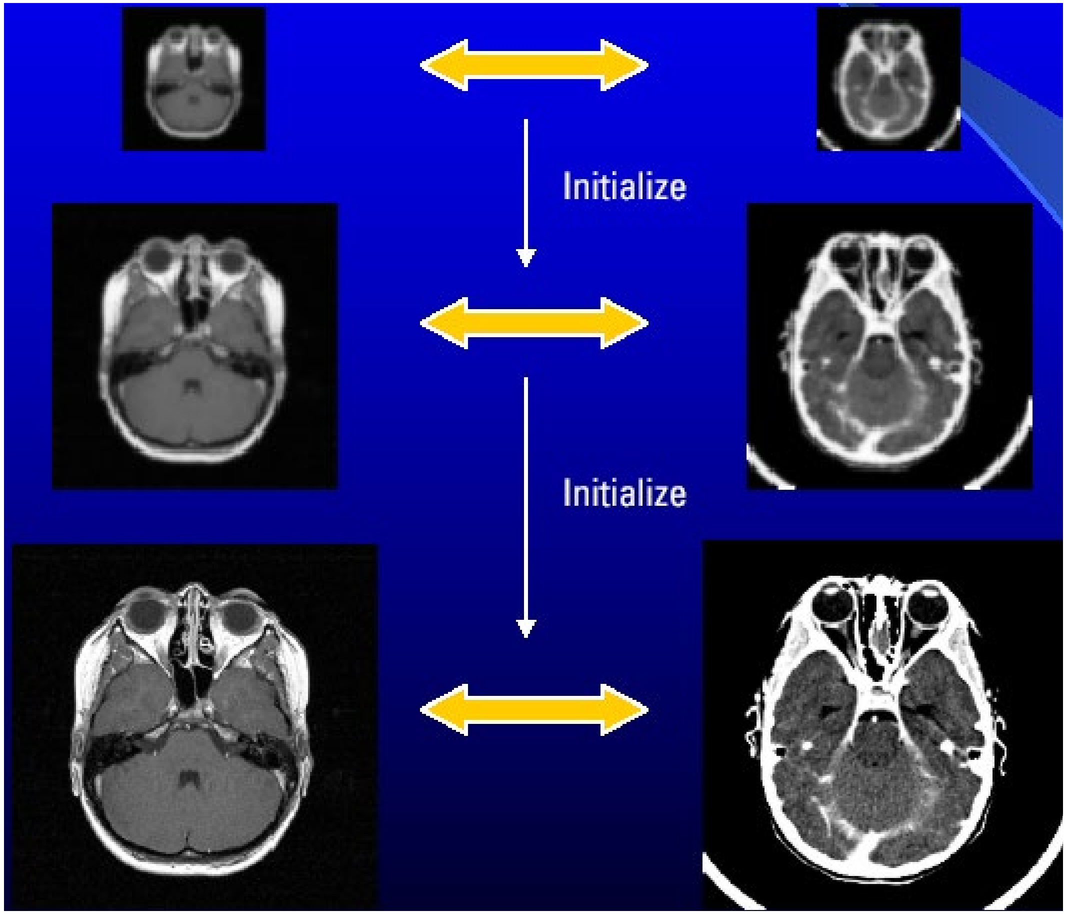
# Refinement of control point location

- Mutual information method



# Pyramidal representation

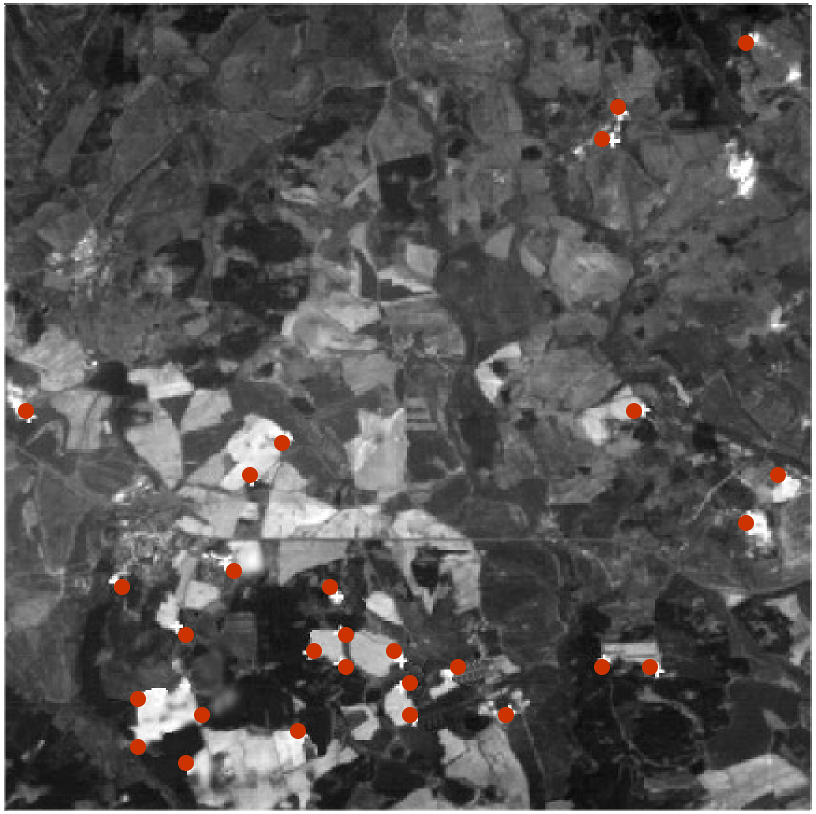
## Processing from coarse to fine level

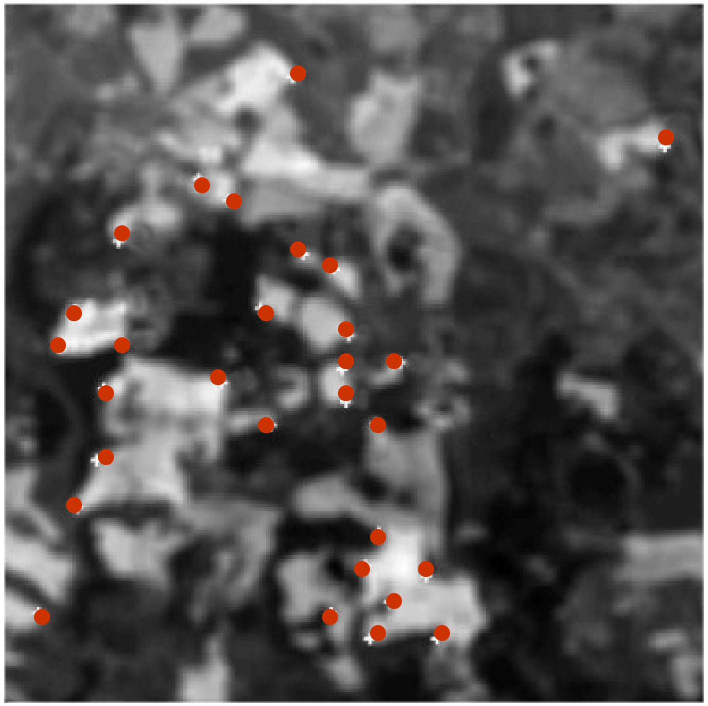


# Feature-based methods

- Combinatorial matching (no feature description). Graph matching, parameter clustering. Global information only is used.
- Matching in the feature space (pattern classification). Local information only is used.
- Hybrid matching (combination of both to get higher robustness)

# Matching in the feature space





$$\min_{k,m} \text{distance}((v1_k, v2_k, v3_k, \ldots), (\overline{v1}_m, \overline{v2}_m, \overline{v3}_m, \ldots))$$

# Mapping function design

- **Global functions**

  **Similarity, affine, projective transform**

  **Low-order polynomials**

# Similarity transform – Least square fit

**translation [$\Delta x$, $\Delta y$], rotation $\varphi$, uniform scaling $s$**

$$x' = s\,(x * \cos\varphi - y * \sin\varphi) + \Delta x$$
$$y' = s\,(x * \sin\varphi + y * \cos\varphi) + \Delta y$$

$$s\ \cos\varphi = a, \qquad s\ \sin\varphi = b$$

$$\min\left(\Sigma_{i=1}\{[\,x_i' - (ax_i - by_i) - \Delta x\,]^2 + [\,y_i' - (bx_i + ay_i) - \Delta y\,]^2\}\right)$$

$$\begin{vmatrix} \Sigma(x_i^2 + y_i^2) & 0 & \Sigma x_i & \Sigma y_i \\ 0 & \Sigma(x_i^2 + y_i^2) & -\Sigma y_i & \Sigma x_i \\ \Sigma x_i & -\Sigma y_i & N & 0 \\ \Sigma y_i & \Sigma x_i & 0 & N \end{vmatrix} \bullet \begin{vmatrix} a \\ b \\ \Delta x \\ \Delta y \end{vmatrix} = \begin{vmatrix} \Sigma(x_i' x_i - y_i' y_i) \\ \Sigma(y_i' x_i - x_i' y_i) \\ \Sigma x_i' \\ \Sigma y_i' \end{vmatrix}$$

# Transform models

- Global functions

  Similarity, affine, projective transform

  Low-order polynomials

- Local functions

  Piecewise affine, piecewise cubic

  Thin-plate splines

  Radial basis functions

# A motivation to TPS

- Unconstrained interpolation – ill posed
- Constrained interpolation

$$\min \iint \left(\frac{\partial^2 f}{\partial x \partial x}\right)^2 + 2\left(\frac{\partial^2 f}{\partial x \partial y}\right)^2 + \left(\frac{\partial^2 f}{\partial y \partial y}\right)^2 dxdy$$

**This task has an analytical solution - TPS**

# TPS

$$f(x,y) = a_1 + a_x x + a_y y + \sum_{i=1}^{p} w_i U\left(\|(x_i, y_i) - (x,y)\|\right)$$

$$U(r) = r^2 \log r$$

# Computing the TPS coefficients

$$\begin{bmatrix} K & P \\ P^T & O \end{bmatrix} \begin{bmatrix} w \\ a \end{bmatrix} = \begin{bmatrix} v \\ o \end{bmatrix}$$

$$K_{ij} = U(\|(x_i, y_i) - (x_j, y_j)\|)$$

$$\sum_{i=1}^{p} w_i = 0 \quad \text{and}$$

$$\sum_{i=1}^{p} w_i x_i = \sum_{i=1}^{p} w_i y_i = 0 \ .$$

# Approximating TPS

- Regularized approximation – well posed

$$\min \; J(f) = E(f) + b\, R(f)$$

E(f) - error term

R(f) - regularization term

b - regularization parameter

# The choice of E and R

$$E(f) = \sum (x_i' - f(x_i,y_i))^2$$

$$R(f) = \iint \left(\frac{\partial^2 f}{\partial x \partial x}\right)^2 + 2\left(\frac{\partial^2 f}{\partial x \partial y}\right)^2 + \left(\frac{\partial^2 f}{\partial y \partial y}\right)^2 dx dy$$

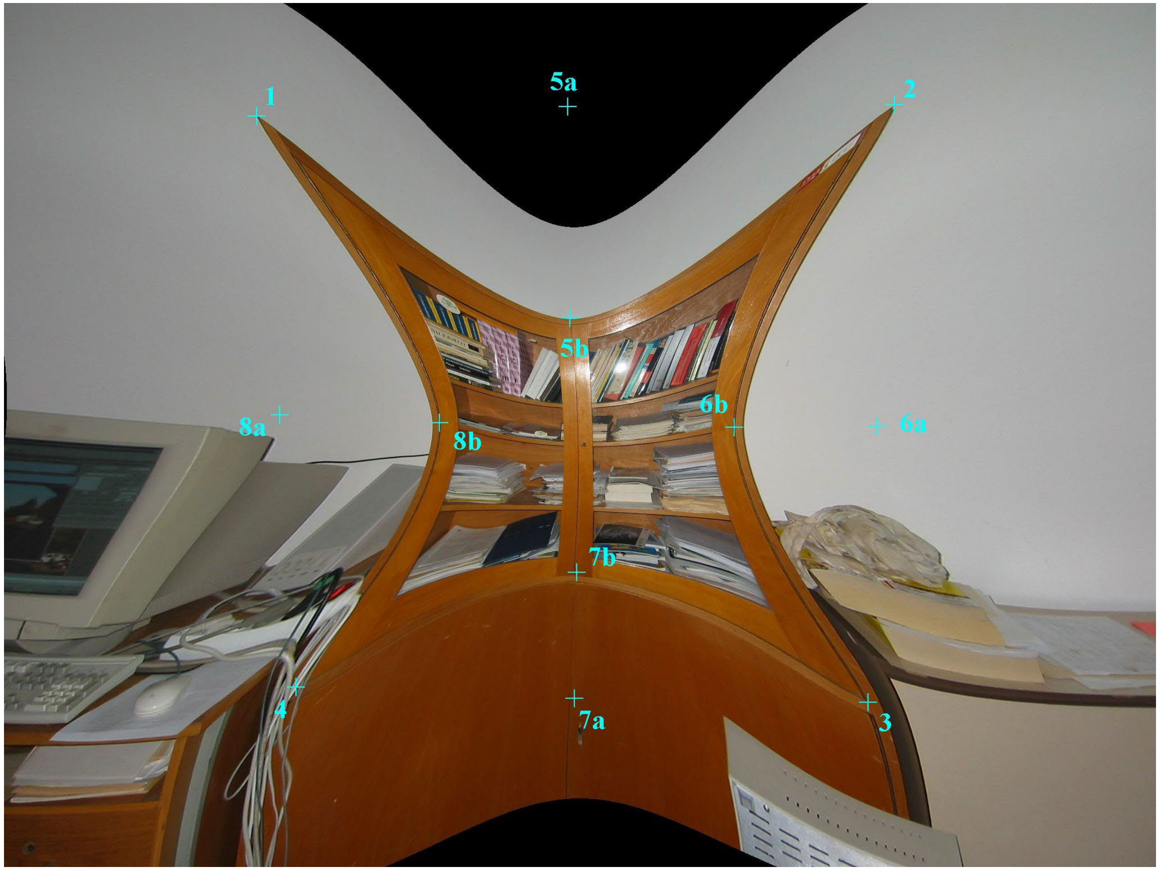
**The solution of the same form - “smoothing” TPS**

# Computing the smoothing TPS coefficients

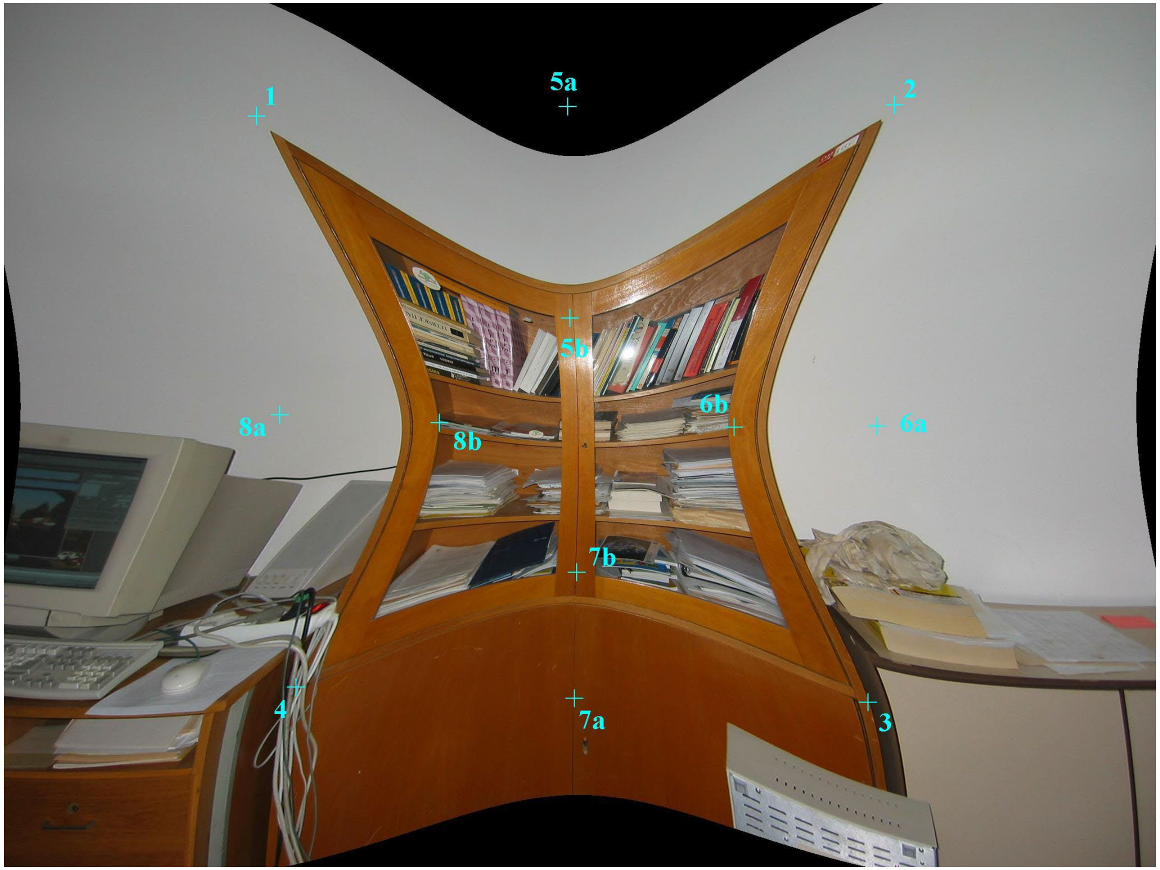
replacing the matrix $K$ by $K + \lambda I$

# The role of parameter *b*

# TPS, 1 >> b

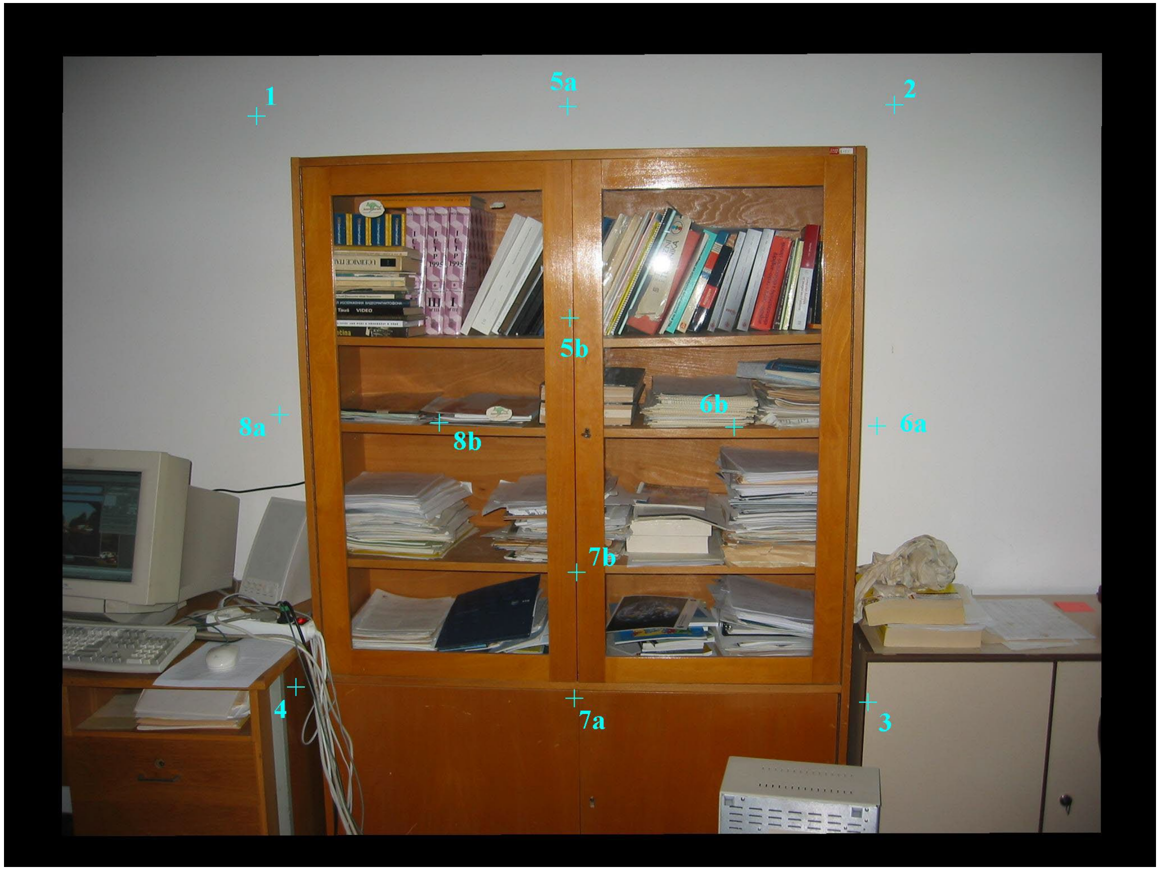


# TPS, 1 > b



# TPS, 1 < b

# TPS, 1 << b



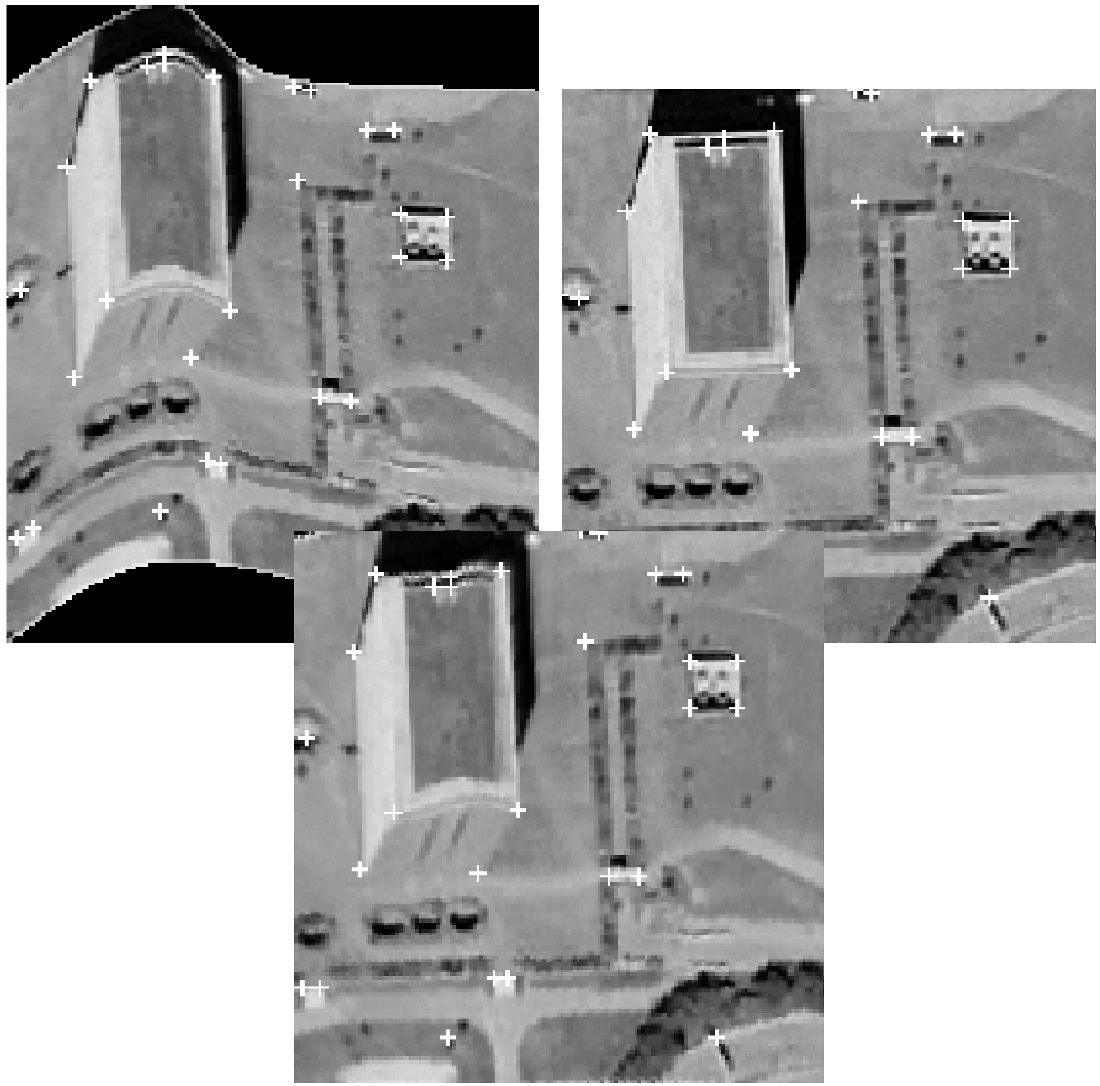
# The original

## The choice of "optimal" parameter *b*

- "Leave-one-out" cross-validation
- Minimizing the mean square error over *b*
- Possibly unstable

# TPS registration

deformed　　　　reference



# 3D shape recovery by TPS

# The drawback: evaluation of TPS is slow

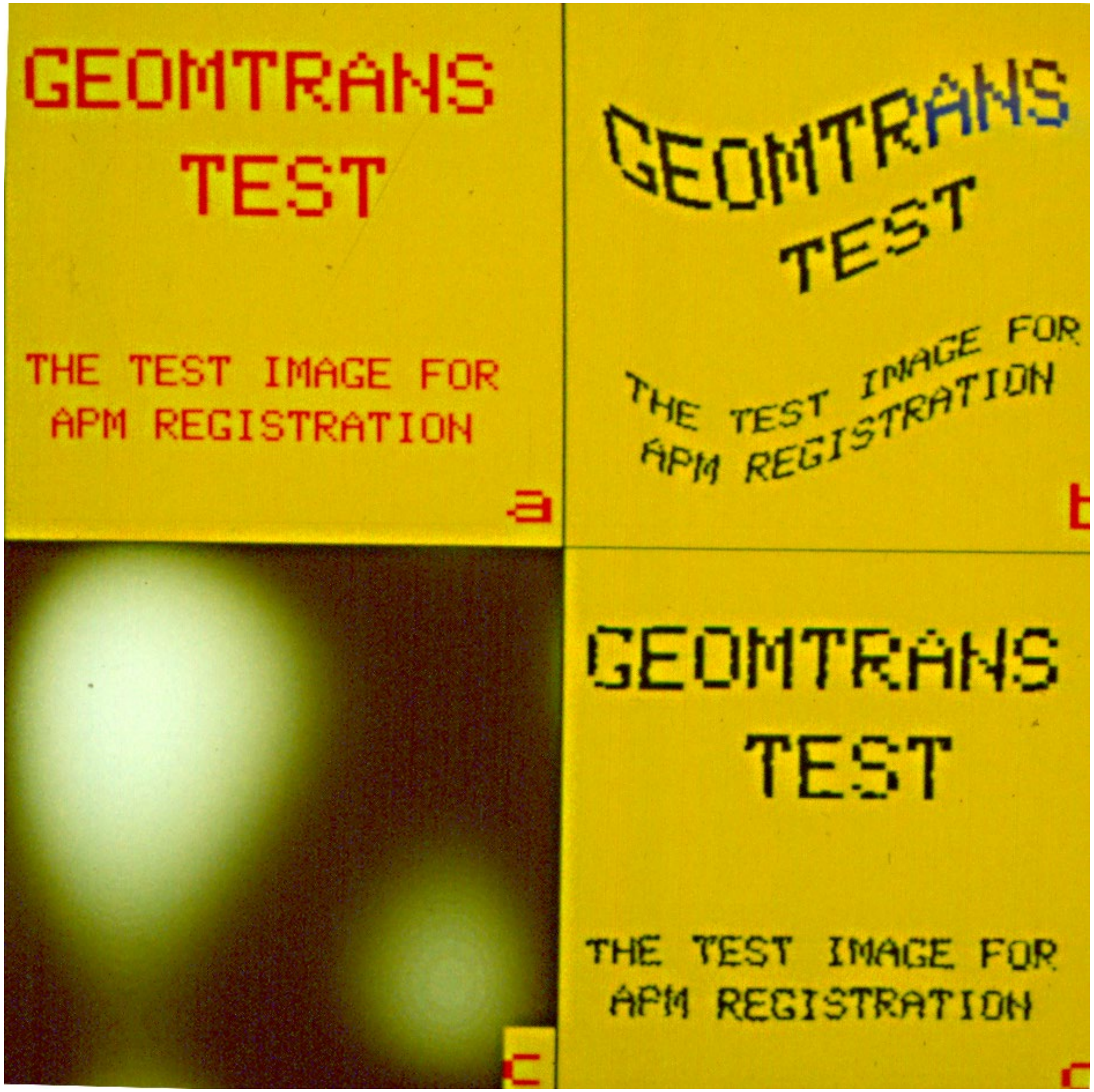
## Speed-up techniques

- Adaptive piecewise approximation
- Subtabulation schemes (Powell)
- Approximation by power series

# Adaptive piecewise approximation

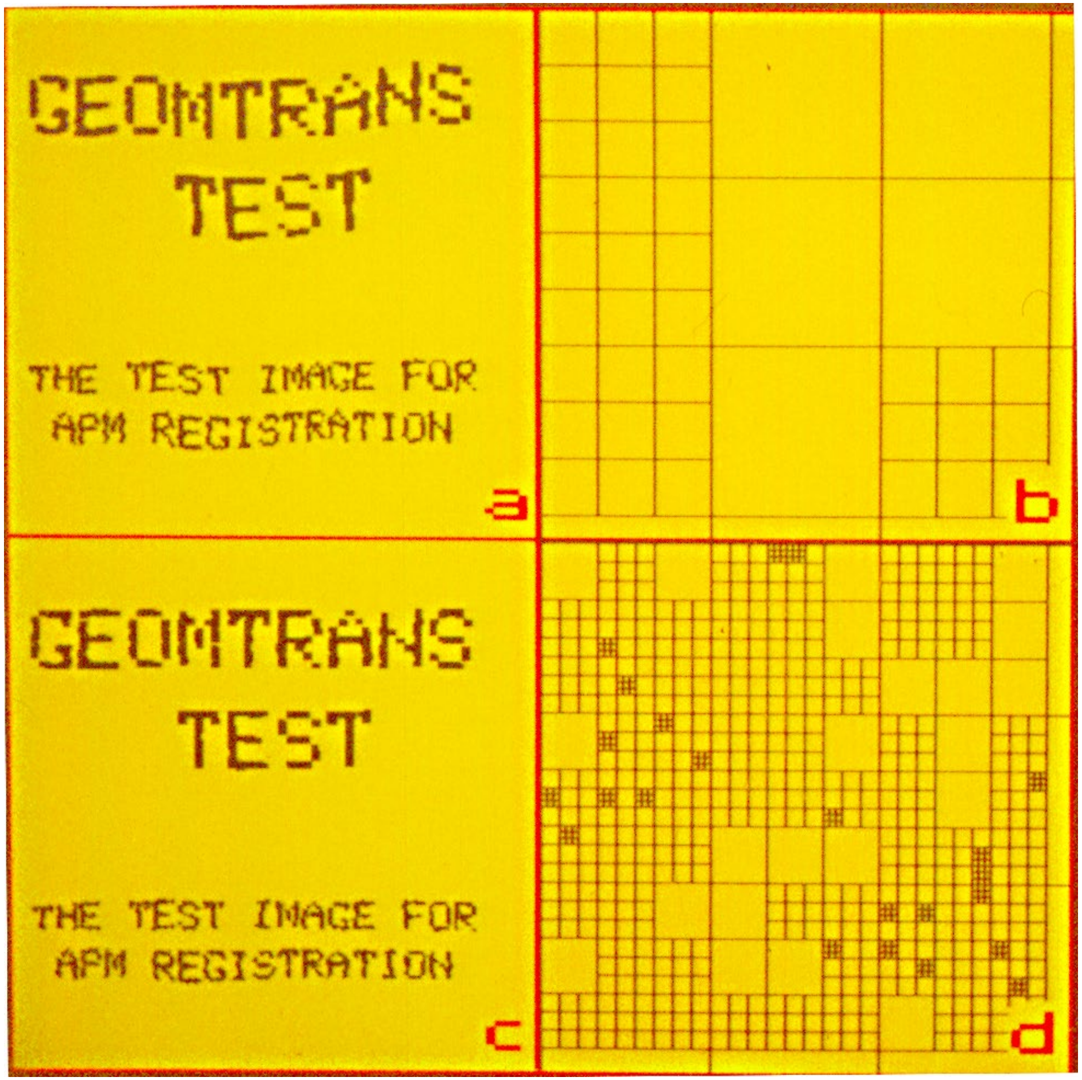
- Image decomposition according to the distortion (quadtree, bintree, ... )
- Transformation of each block by affine or projective transform

# Adaptive piecewise approximation




TPS

# Adaptive piecewise approximation



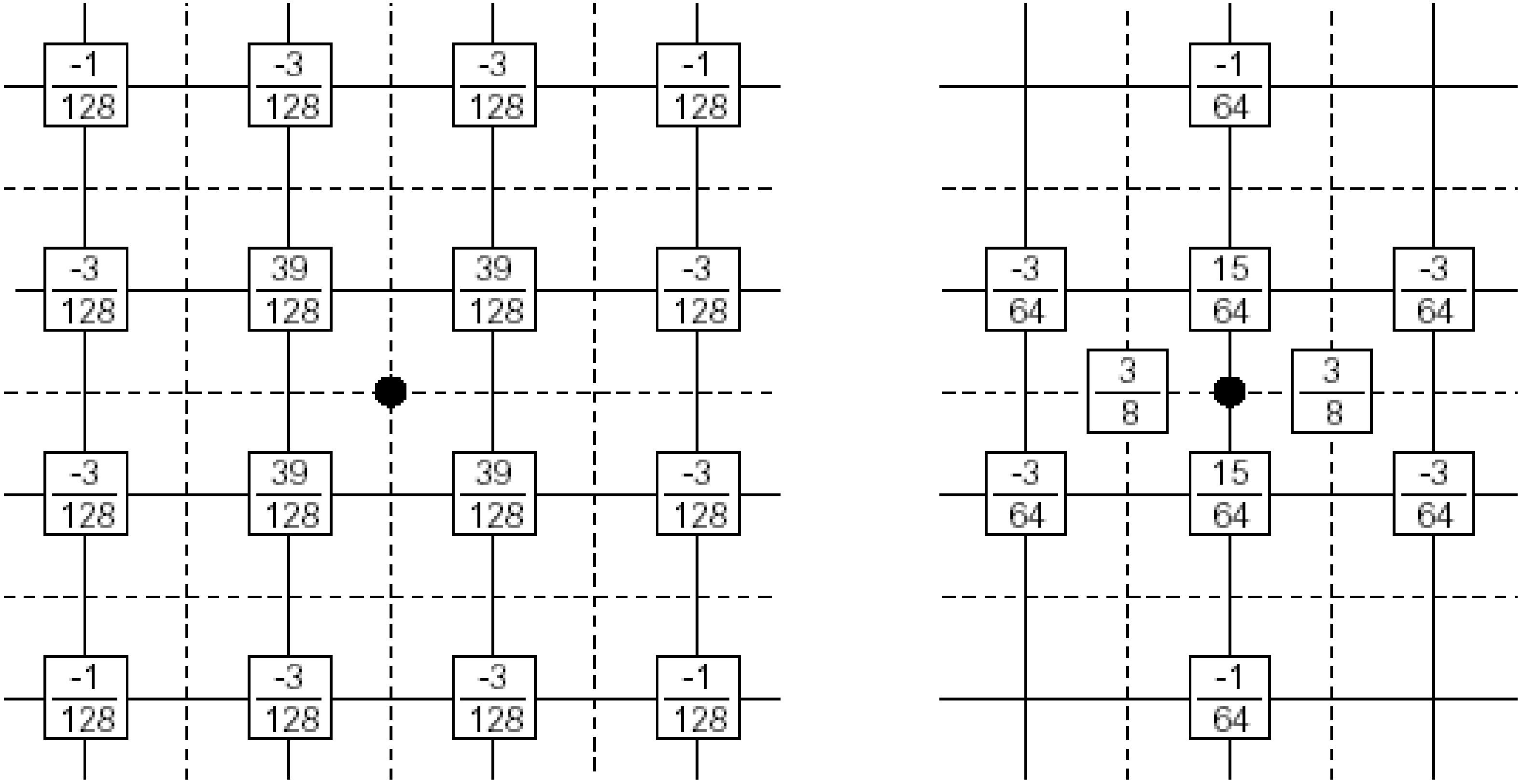

**Piecewise projective**

# Adaptive piecewise approximation

# Subtabulation schemes

- Calculate the TPS values on a coarse grid
- Refine the grid twice
- Approximate the missing TPS values

# Powell's scheme



# Luner's scheme