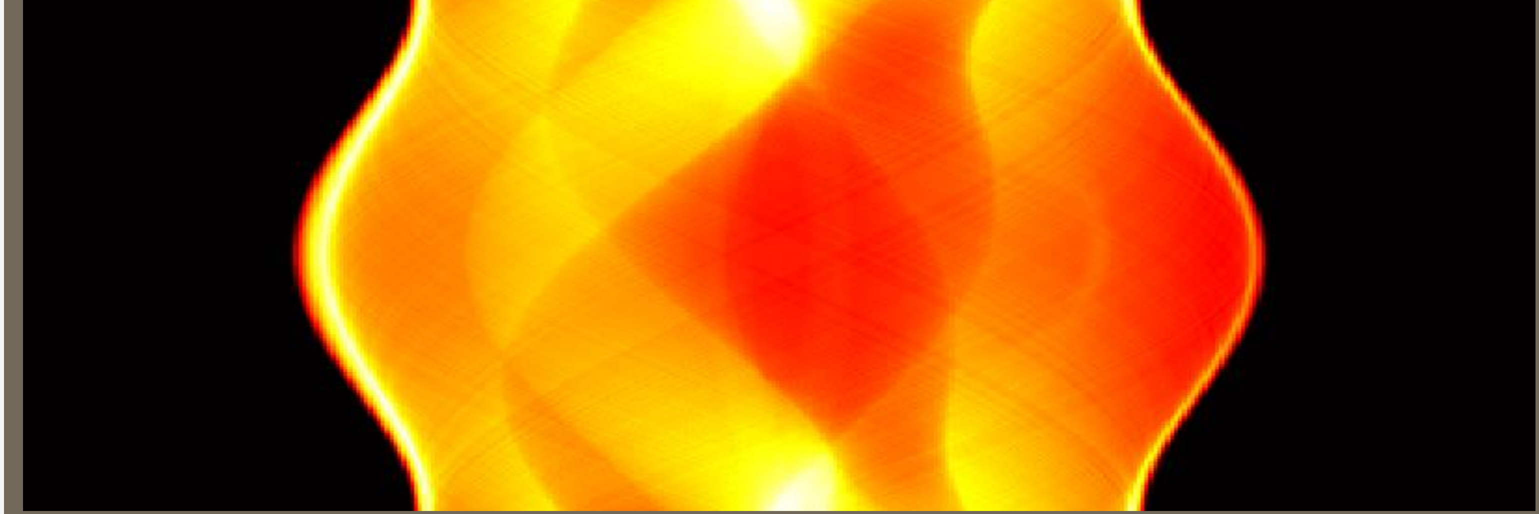


Radon & Hough & Fourier Transform

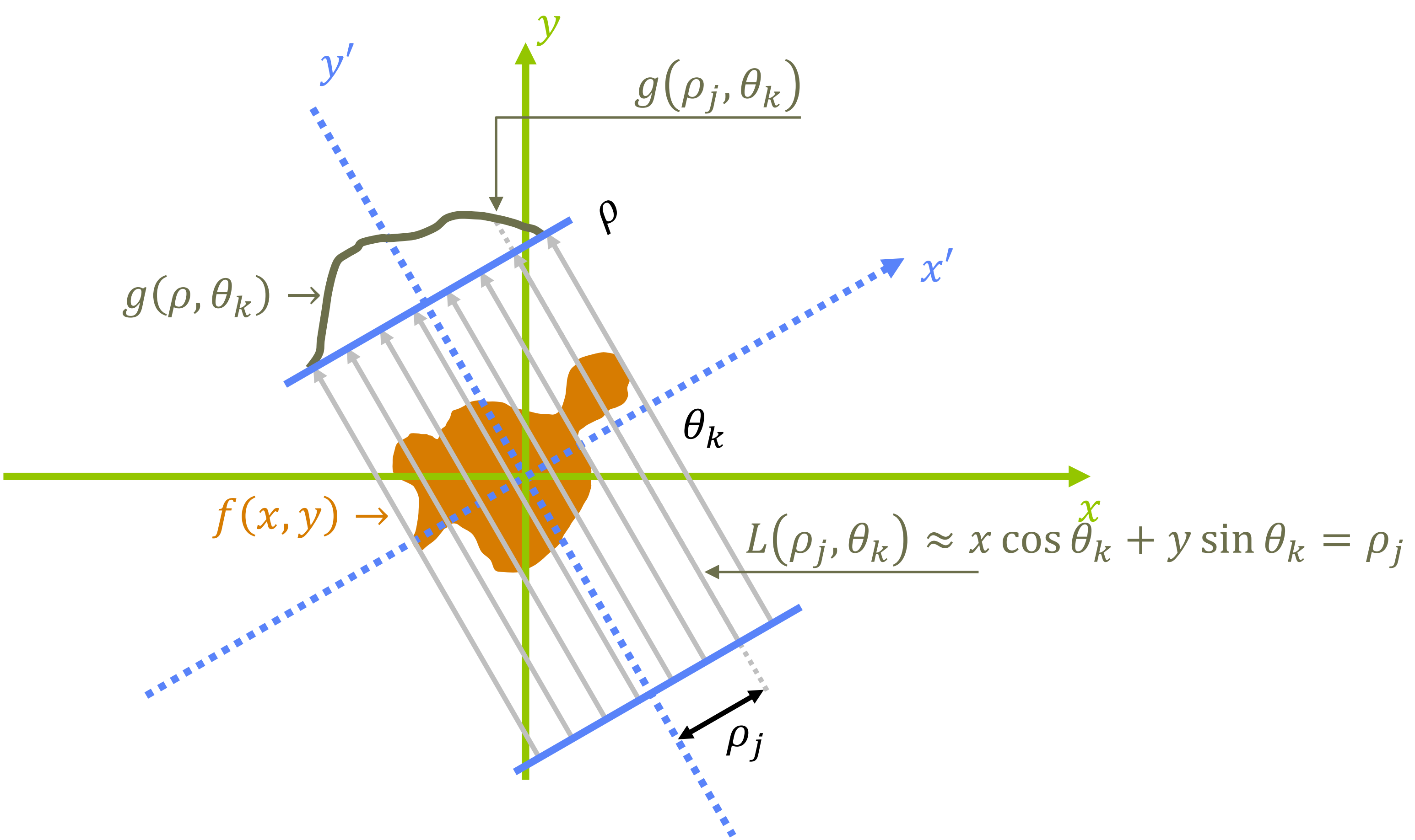
ZOI – UTIA

Adam Novozámský (novozamsky@utia.cas.cz)

Radon

[Radon 1917]

$$g(\rho_j, \theta_k) = \iint_{-\infty}^{\infty} f(x,y)\delta(x\cos\theta_k + y\sin\theta_k - \rho_j)dxdy$$



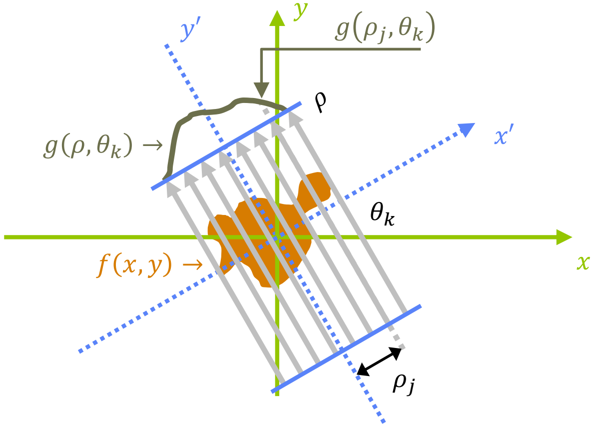
Radon

[Radon 1917]

$$g(\rho_j,\theta_k) = \iint_{-\infty}^{\infty} f(x,y)\delta\left(x\cos\theta_k + y\sin\theta_k - \rho_j\right)dxdy$$

$$g(\rho,\theta) = \iint_{-\infty}^{\infty} f(x,y)\delta(x\cos\theta + y\sin\theta - \rho)dxdy$$

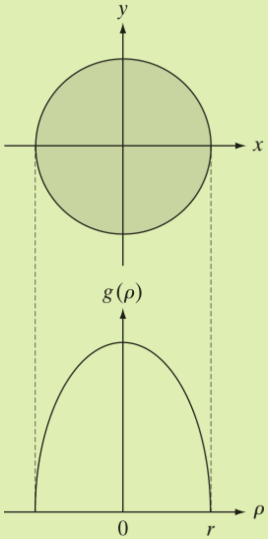
$$g(\rho,\theta) = \sum_{x=0}^{M-1}\sum_{y=0}^{N-1} f(x,y)\delta(x\cos\theta + y\sin\theta - \rho)$$



Illustration

$$f(x,y) = \begin{cases} A & x^2 + y^2 \le r^2 \\ 0 & otherwise \end{cases}$$

$$g(\rho,\theta) = \int_{-\sqrt{r^2-\rho^2}}^{\sqrt{r^2-\rho^2}} A dy = g(\rho) = \begin{cases} 2A\sqrt{r^2-\rho^2} & |\rho| \le r \\ 0 & otherwise \end{cases}$$



Radon

[Radon 1917]

$$g(\rho,\theta)=\sum_{x=0}^{M-1}\sum_{y=0}^{N-1} f(x,y)\delta(x\cos\theta+y\sin\theta-\rho)$$

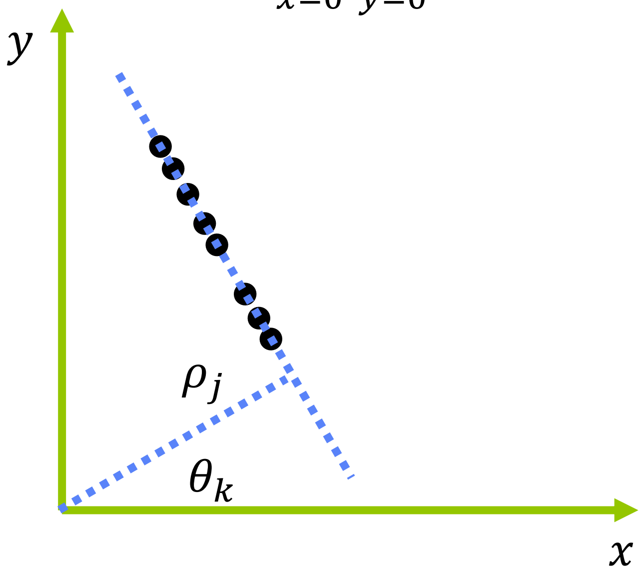
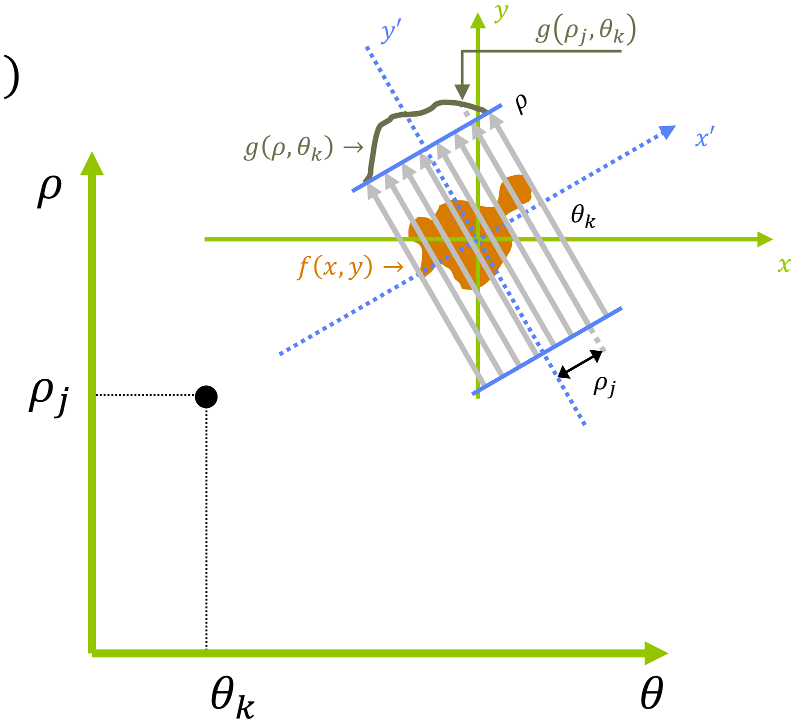


Image space

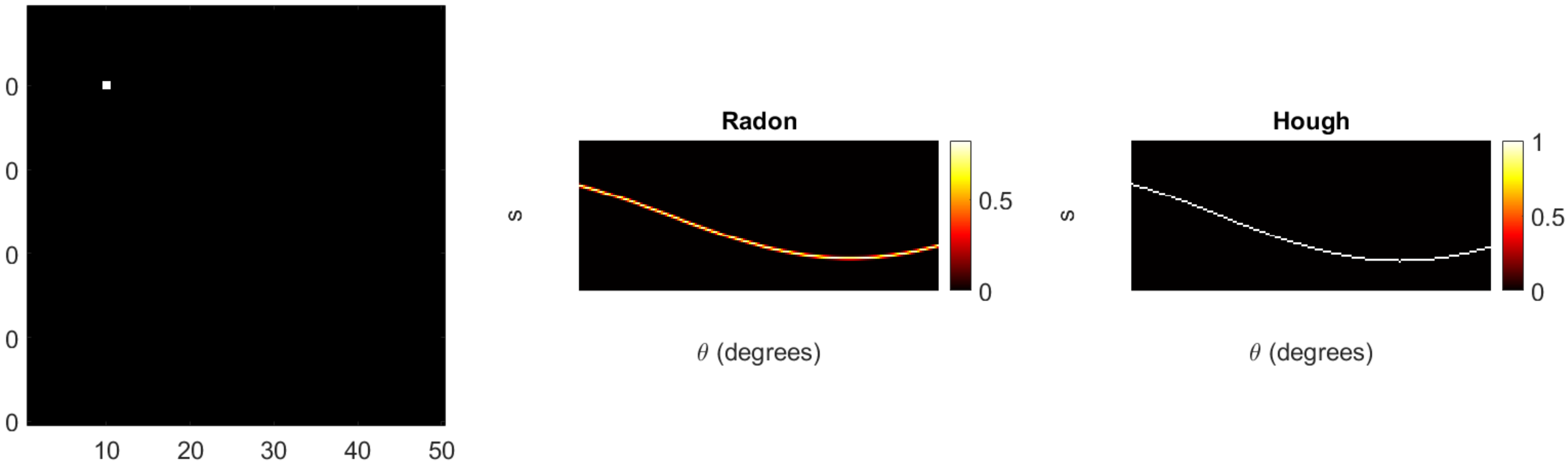


Radon space

- **Property 1:** A point in the picture plane corresponds to a sinusoidal curve in the parameter plane.
- **Property 2:** A point in the parameter plane corresponds to a straight line in the picture plane.
- **Property 3:** Points lying on the same straight line in the picture plane correspond to curves through a common point in the parameter plane.
- **Property 4:** Points lying on the same curve in the parameter plane correspond to lines through the same point in the picture plane.

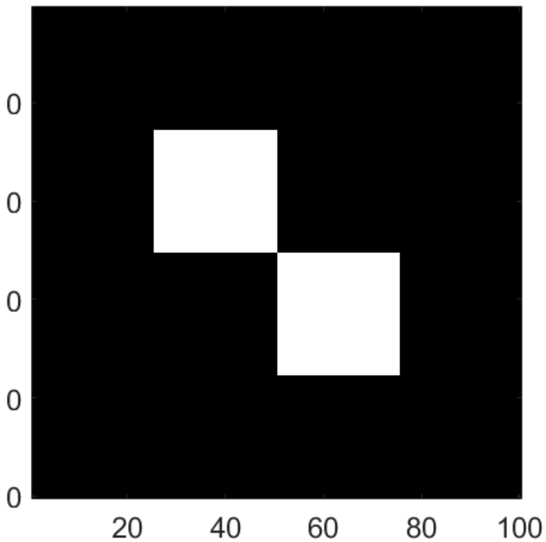
Radon & Hough

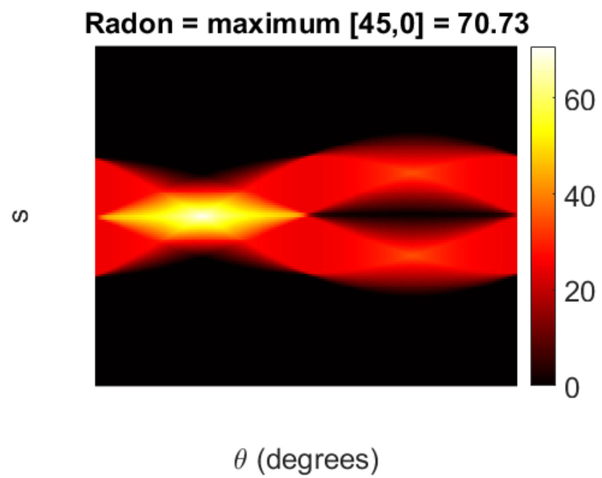
the relationship

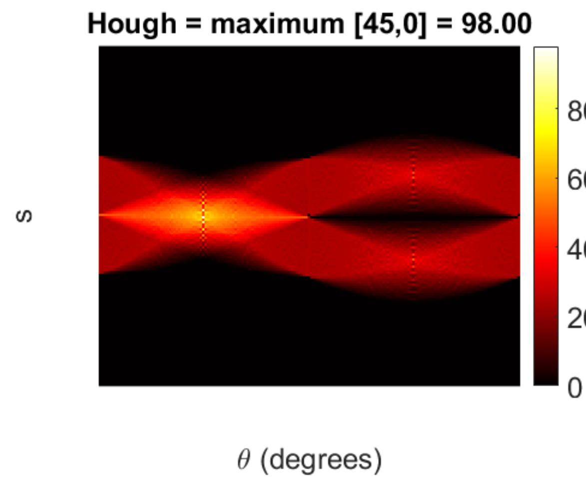


Radon & Hough

the relationship

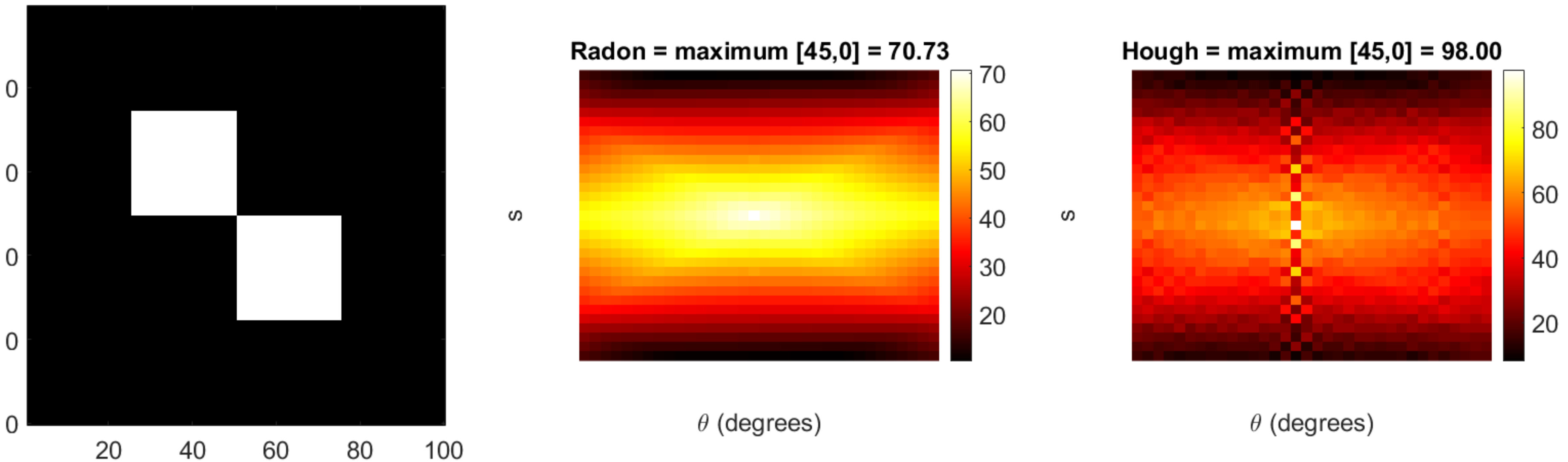






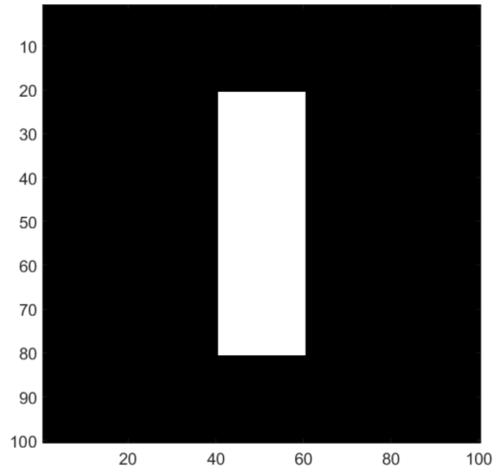
Radon & Hough

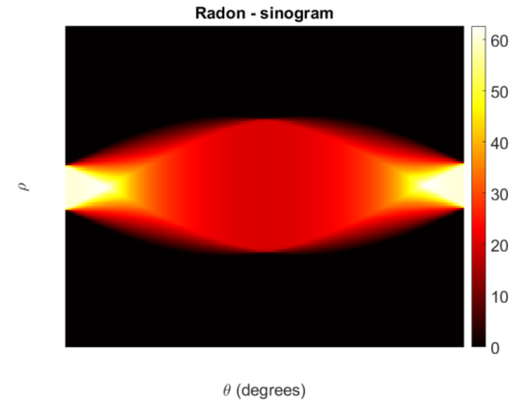
the relationship

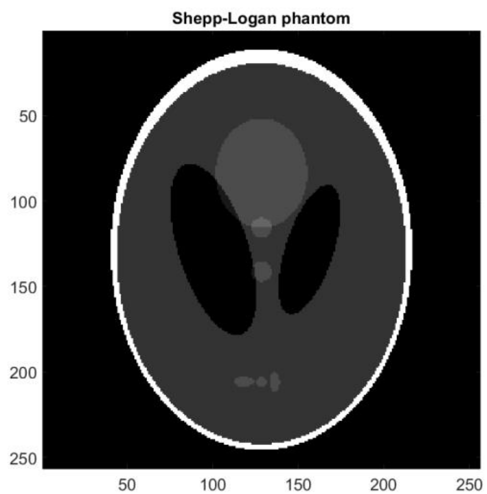


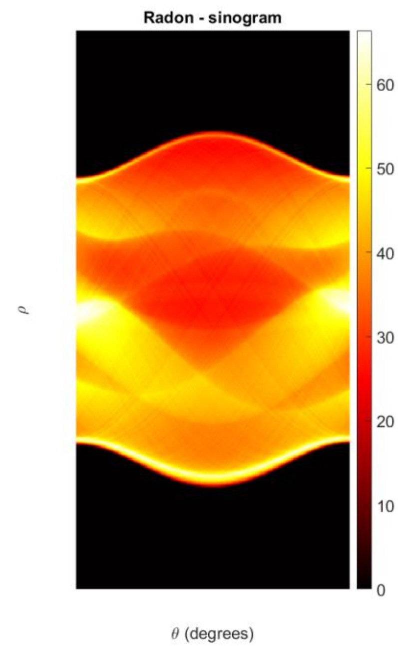
Radon

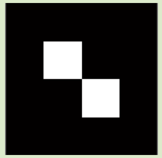
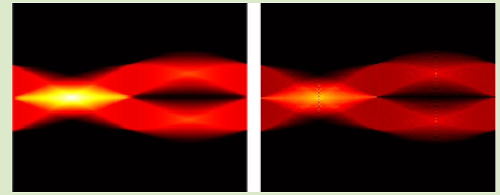
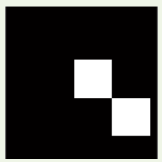
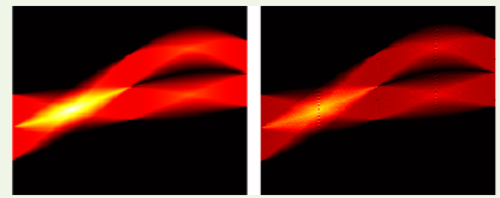
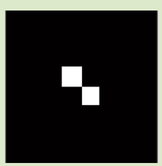
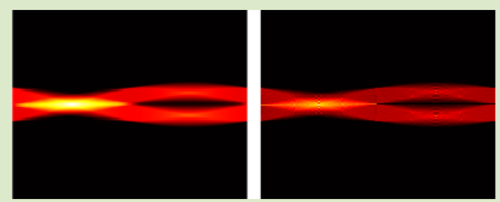
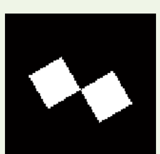
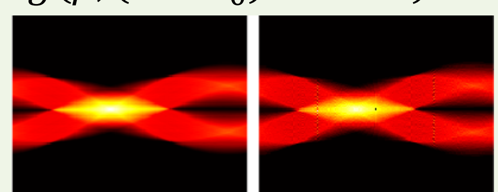
[Radon 1917]





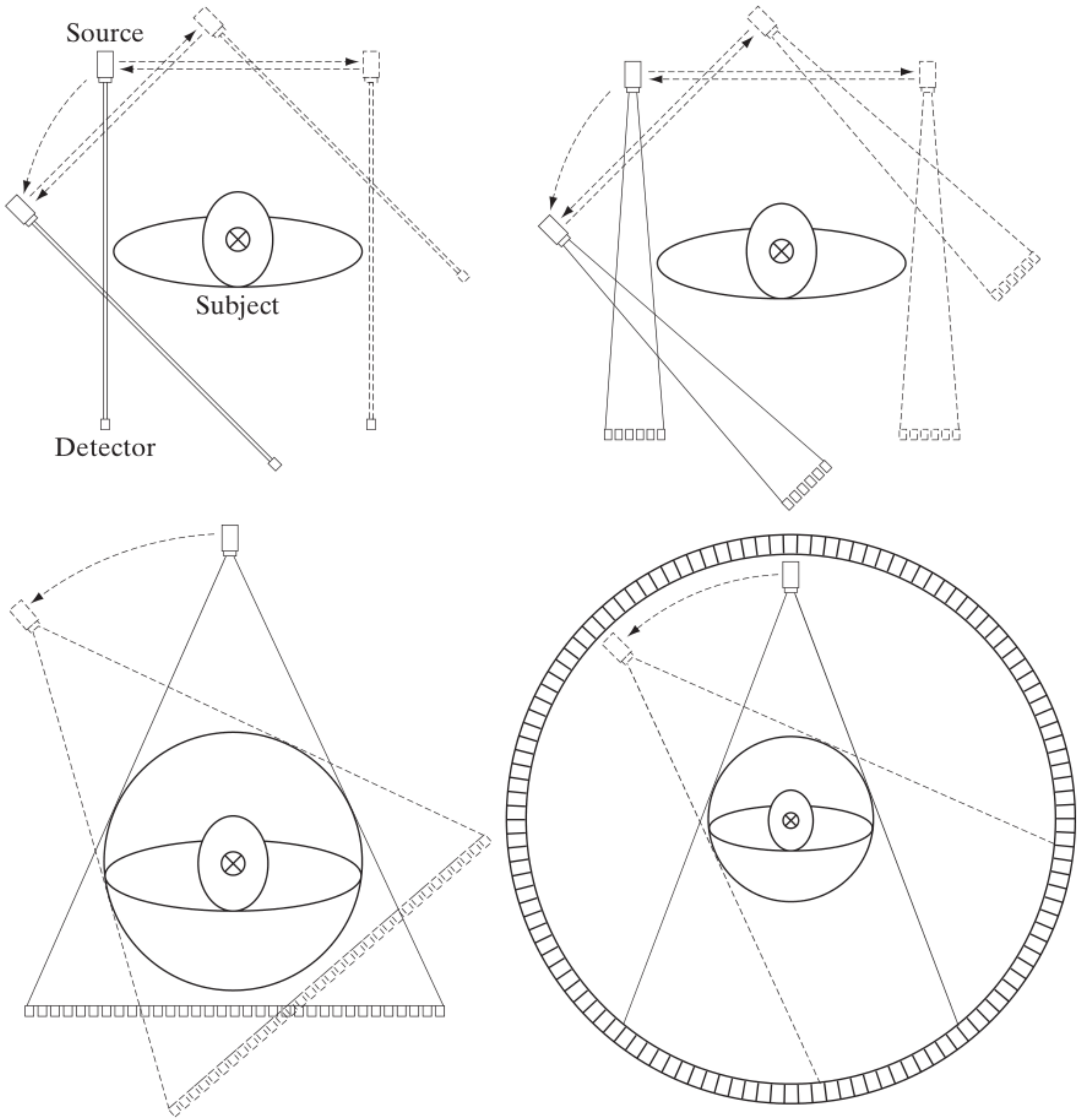




Property	Image space	Radon space & Hough space
Original	$f(x,y)$ 	$g(\rho,\theta)$ 
Translation	$f(x-x_0, y-y_0)$ 	$g(\rho - x_0 \cos\theta - y_0 \sin\theta\,, \theta)$ 
Scaling	$f(\alpha x, \alpha y)$ 	$\frac{1}{\lvert\alpha\rvert} g(\alpha\rho, \theta)$ 
Rotation	$f_{polar}(r, \phi + \theta_0)$ 	$g(\rho, (\theta + \theta_0) \bmod 2\pi)$ 

Radon

[Gonzalez 2008 3rd]



Radon

Back projection: formal interpretation

- for a single point, $g(\rho_j, \theta_k)$, copying the line $L(\rho_j, \theta_k)$ onto the *empty* image with its intensity $g(\rho_j, \theta_k)$
- repeating this process of all values of ρ_j in the projected signal

$$f_{\theta_k}(x, y) = g(\rho, \theta_k) = g(x \cos \theta_k + y \sin \theta_k, \theta_k)$$
$$f_{\theta}(x, y) = g(x \cos \theta + y \sin \theta, \theta)$$

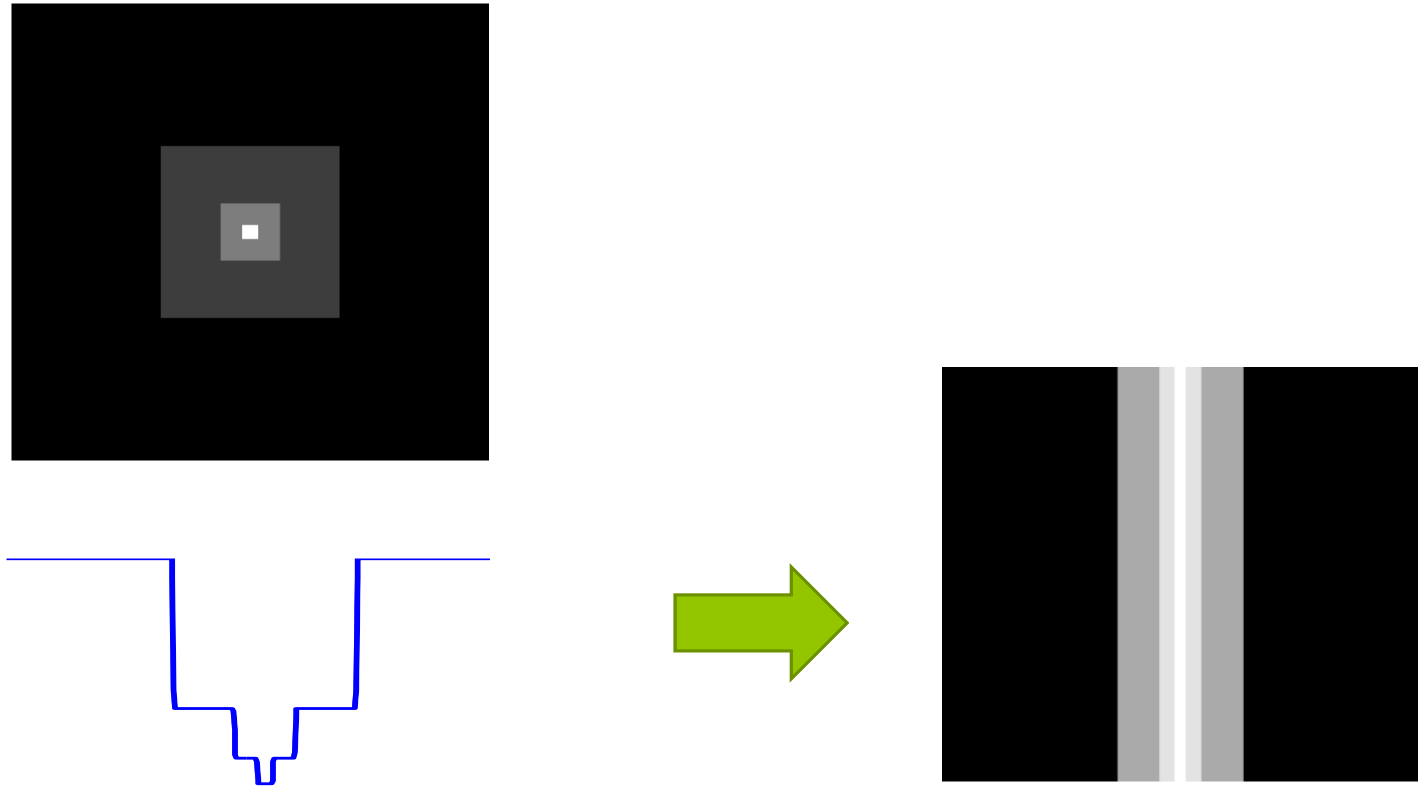
- final image by integrating over all the back-projected images :

$$f(x, y) = \int_0^{\pi} f_\theta(x, y) d\theta \sim \sum_{\theta=0}^{\pi} f_\theta(x, y) \rightarrow laminogram$$

Radon

Back projection:

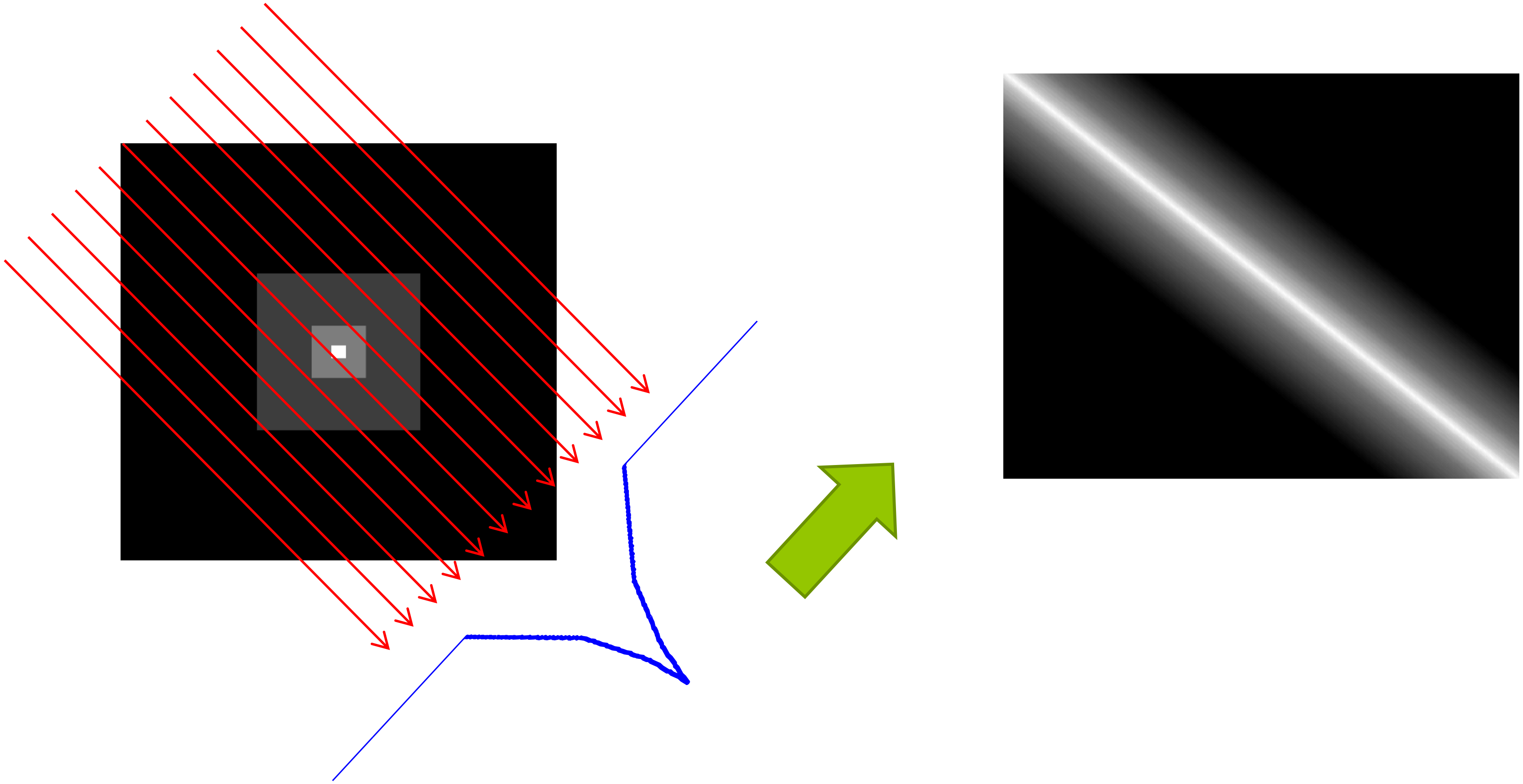
A little trick that almost works!



Radon

Back projection:

A little trick that almost works!



Radon

[Gonzalez 2008 3rd]

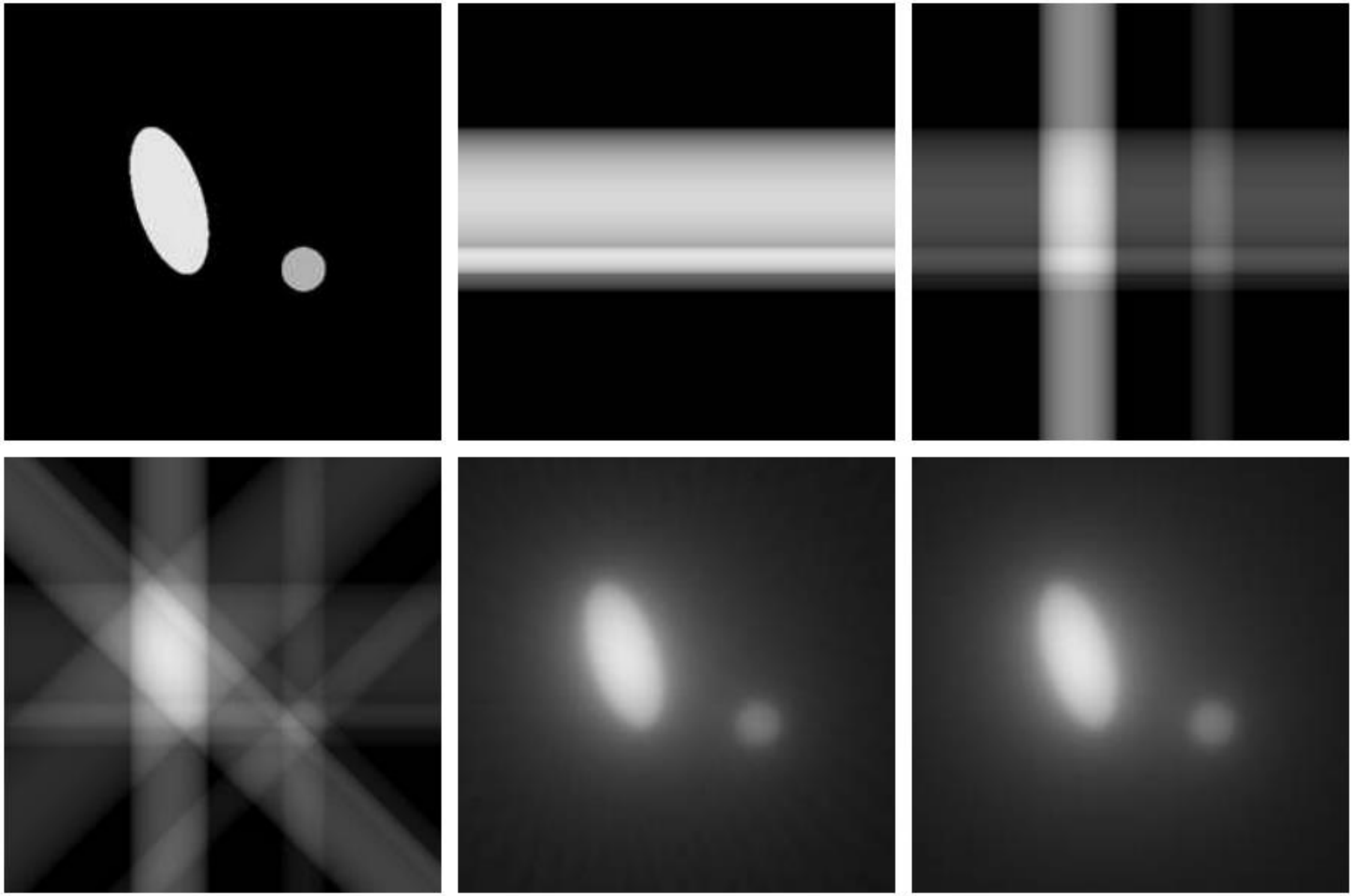
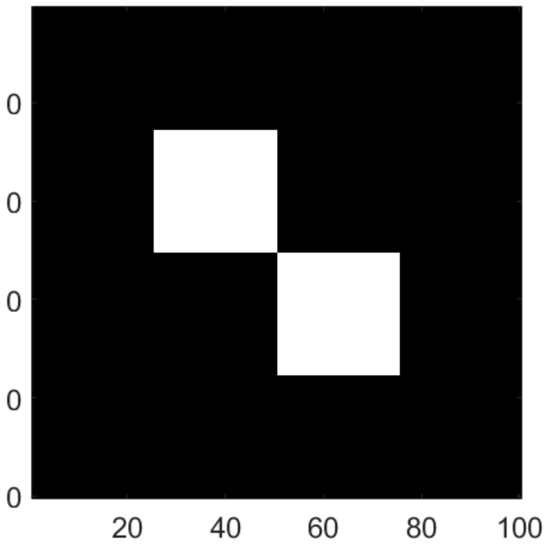
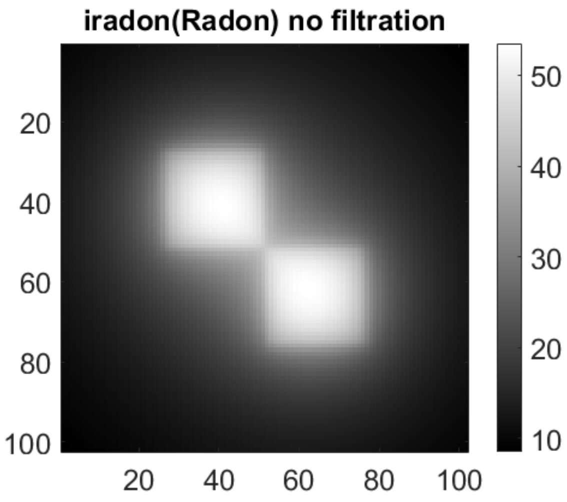


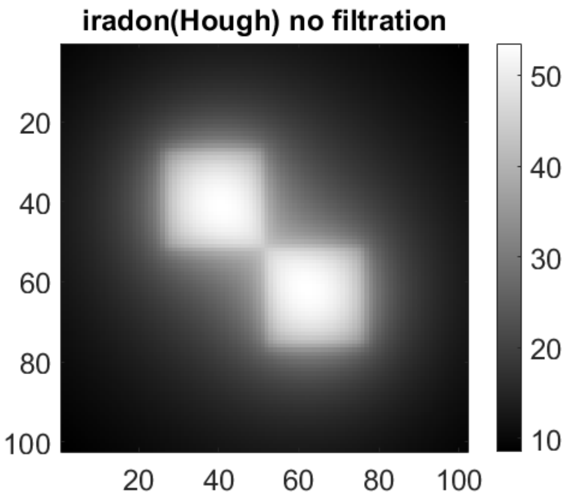
Image	1	2
4	32	64

Radon & Hough

Naive backprojection *(without filtration)*

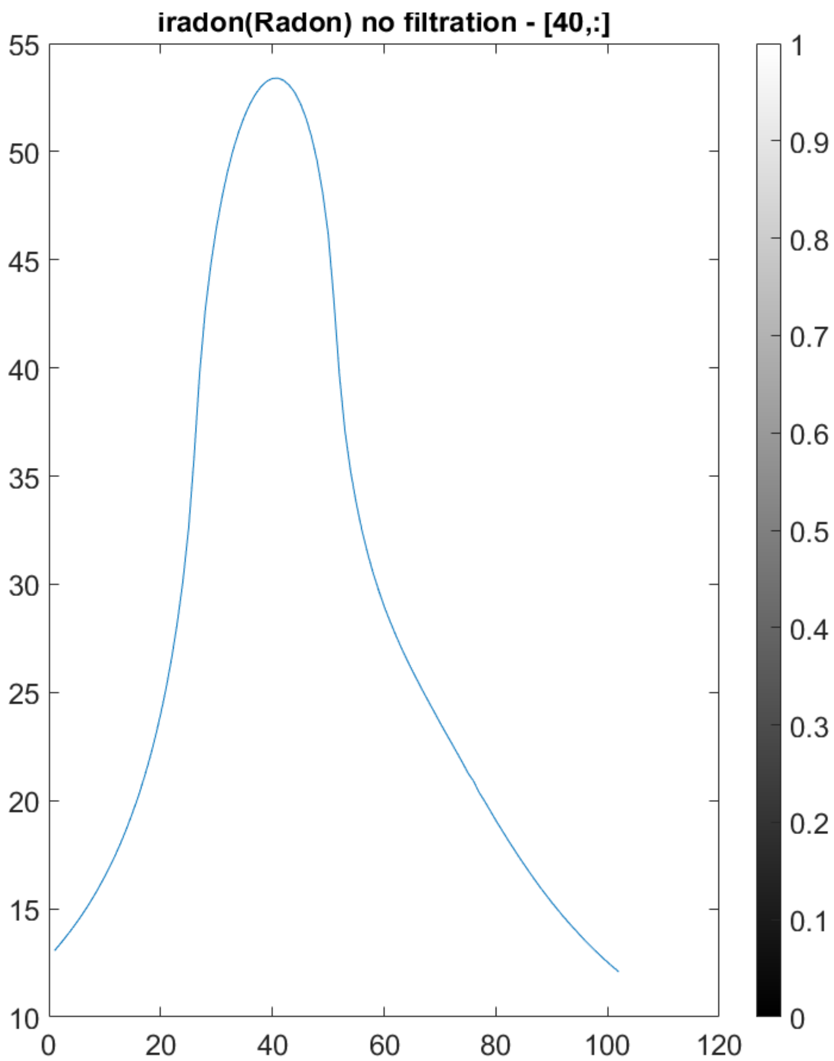


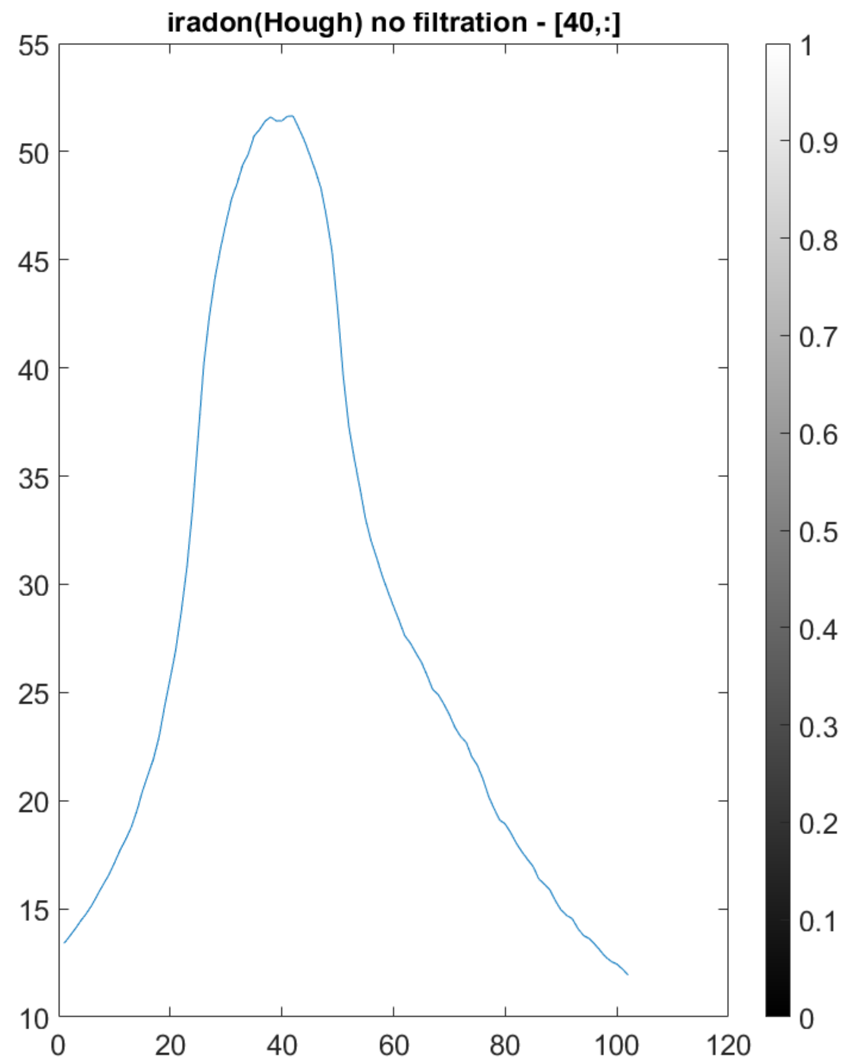




Radon & Hough

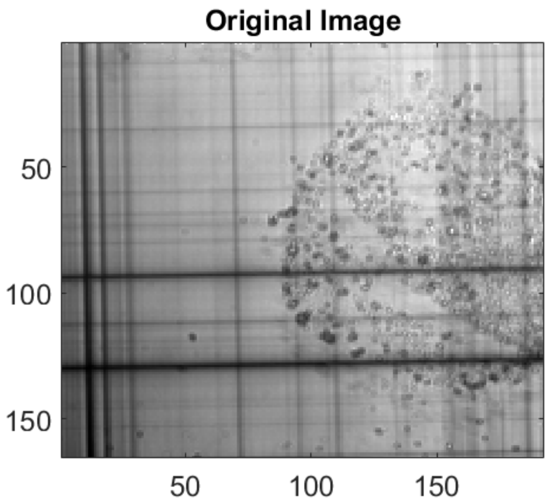
Naive backprojection *(without filtration)*

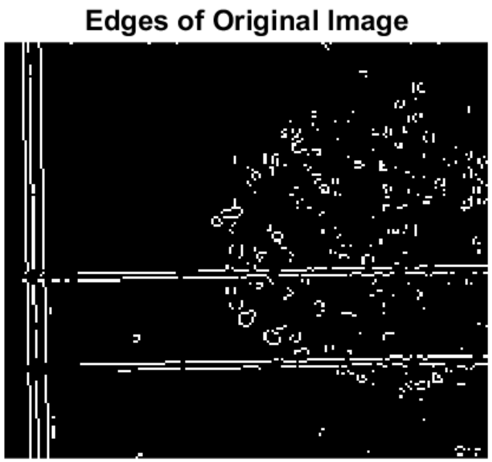


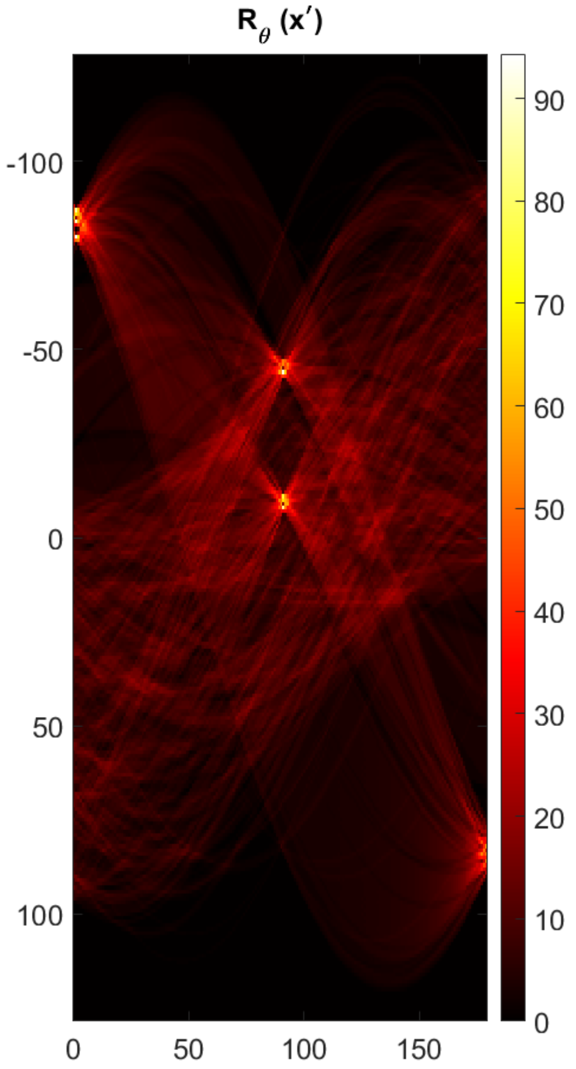


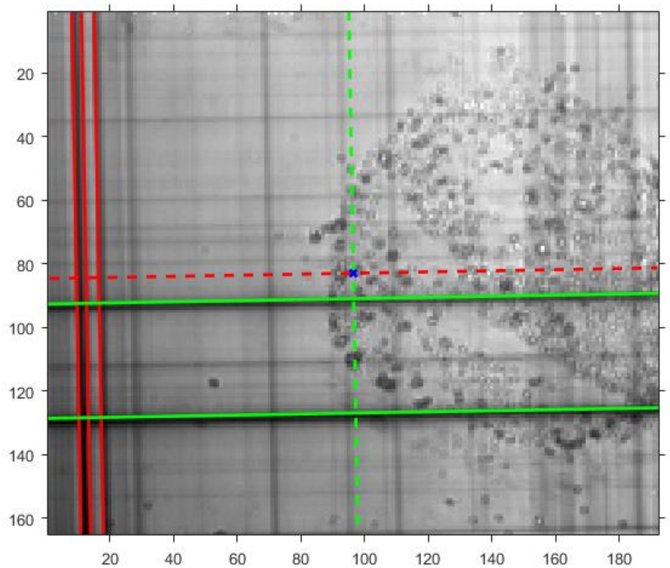
Radon & Hough

line detection

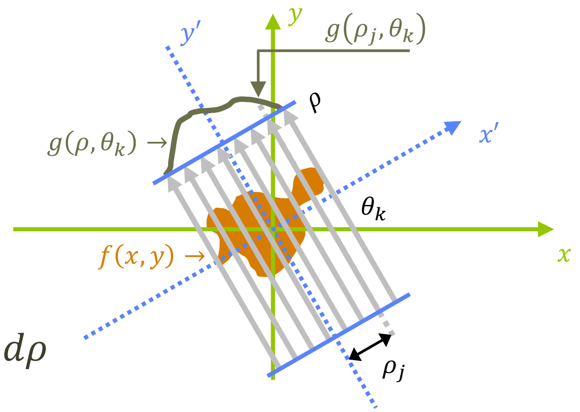








Radon & Fourier



Projection-slice theorem *(Fourier-slice theorem)*

- **1D Fourier transform** of a projection $g(l,\theta)$

$$\mathcal{F}_{1D}\{g(\rho,\theta)\} = G(\omega,\theta) = \int_{-\infty}^{\infty} g(\rho,\theta)e^{-i2\pi\omega\rho}d\rho$$

- **Substitution**

$$g(\rho,\theta) = \iint_{-\infty}^{\infty} f(x,y)\delta(x\cos\theta + y\sin\theta - \rho)dxdy$$

$$G(\omega,\theta) = \iiint_{-\infty}^{\infty} f(x,y)\delta(x\cos\theta + y\sin\theta - \rho)e^{-i2\pi\omega\rho}dxdyd\rho$$

- **Rearranging**

$$G(\omega,\theta) = \iint_{-\infty}^{\infty} f(x,y)\left\{\int_{-\infty}^{\infty}\delta(x\cos\theta + y\sin\theta - \rho)e^{-i2\pi\omega\rho}d\rho\right\}dxdy$$

- **Applying the properties of the delta function**

$$G(\omega,\theta) = \iint_{-\infty}^{\infty} f(x,y)e^{-i2\pi\omega[x\cos\theta + y\sin\theta]}dxdy$$

$$G(\omega,\theta) = \iint_{-\infty}^{\infty} f(x,y)e^{-i2\pi[x\omega\cos\theta + y\omega\sin\theta]}dxdy$$

Radon & Fourier

Projection-slice Theorem

$$G(\omega,\theta)=\iint_{-\infty}^{\infty} f(x,y)e^{-i2\pi[x\omega\cos\theta+y\omega\sin\theta]}dxdy$$

- **This looks like 2D Fourier transform of** $f(x,y)$

$$\mathcal{F}_{2D}\{f(x,y)\}=F(u,v)=\iint_{-\infty}^{\infty} f(x,y)e^{-i2\pi[xu+yv]}dxdy$$

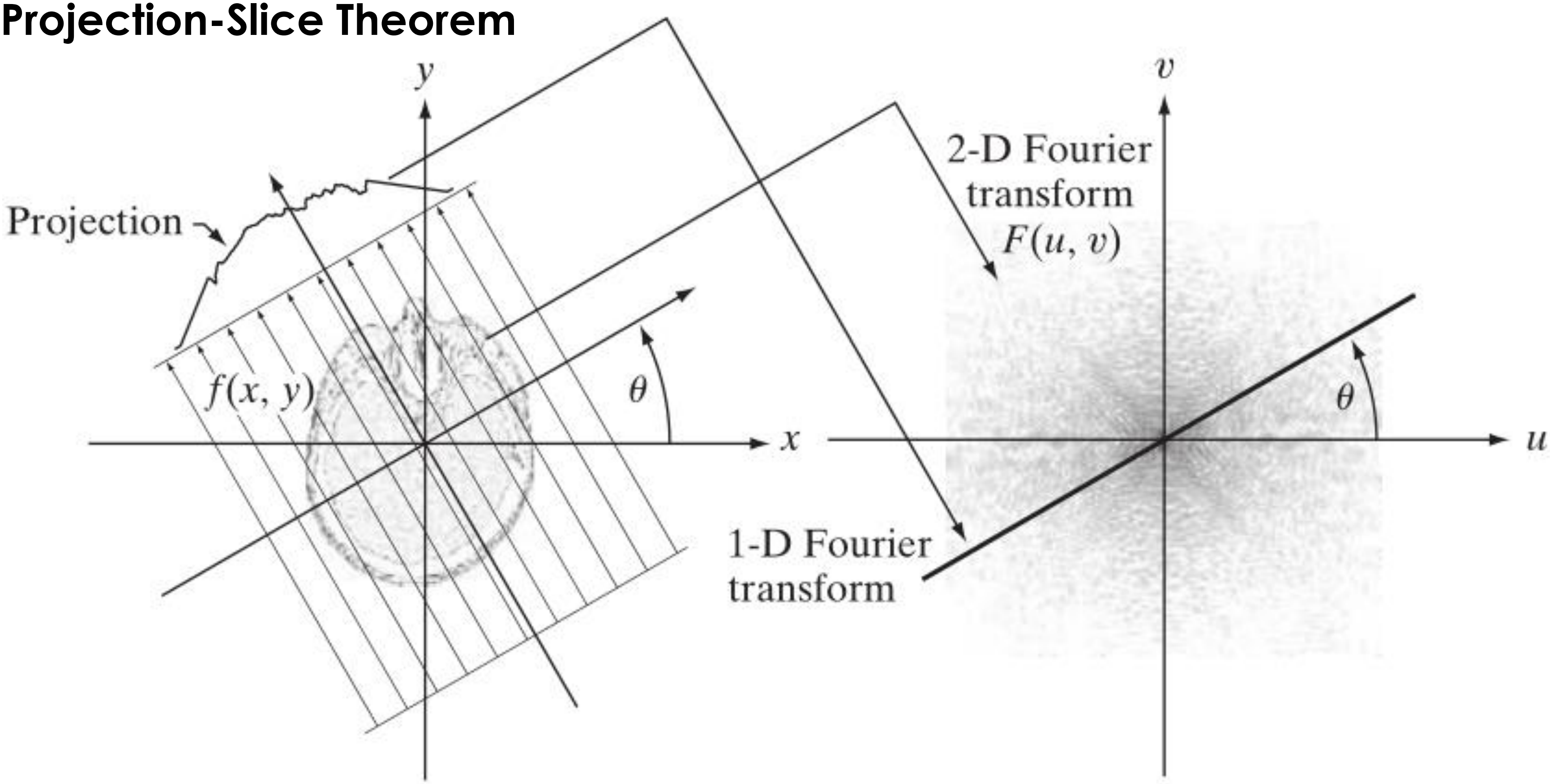
where:

$$u=\omega\cos\theta$$
$$v=\omega\sin\theta$$

$$\Rightarrow\quad G(\omega,\theta)=F(\omega\cos\theta\,,\omega\sin\theta)$$

Radon & Fourier

Projection-Slice Theorem



Radon & Fourier

Reconstruction using parallel-beam filtered backprojections

- **2-D inverse Fourier transform**

$$\mathcal{F}_{2D}^{-}\{F(u,v)\} = f(x,y) = \iint_{-\infty}^{\infty} F(u,v) e^{i2\pi[ux+vy]} du dv$$

$$|u = \omega\cos\theta \quad v = \omega\sin\theta \quad dudv = \omega d\omega d\theta|$$

$$f(x,y) = \int_0^{2\pi}\int_0^{\infty} F(\omega\cos\theta\,,\omega\sin\theta)\, e^{i2\pi\omega[x\cos\boldsymbol{\theta}+\boldsymbol{y}\sin\boldsymbol{\theta}]}\,\omega d\omega d\theta$$

- **using Projection-slice theorem**

$$f(x,y) = \int_0^{2\pi}\int_0^{\infty} G(\omega,\theta)\, e^{i2\pi\omega[x\cos\boldsymbol{\theta}+\boldsymbol{y}\sin\boldsymbol{\theta}]}\,\omega d\omega d\theta$$

- **splitting the integral for θ into two intervals and using** $G(\omega,\theta+\pi) = G(-\omega,\theta)$

$$f(x,y) = \int_0^{\pi}\int_{-\infty}^{\infty} |\omega| G(\omega,\theta)\, e^{i2\pi\omega[x\cos\boldsymbol{\theta}+\boldsymbol{y}\sin\boldsymbol{\theta}]}\, d\omega d\theta$$

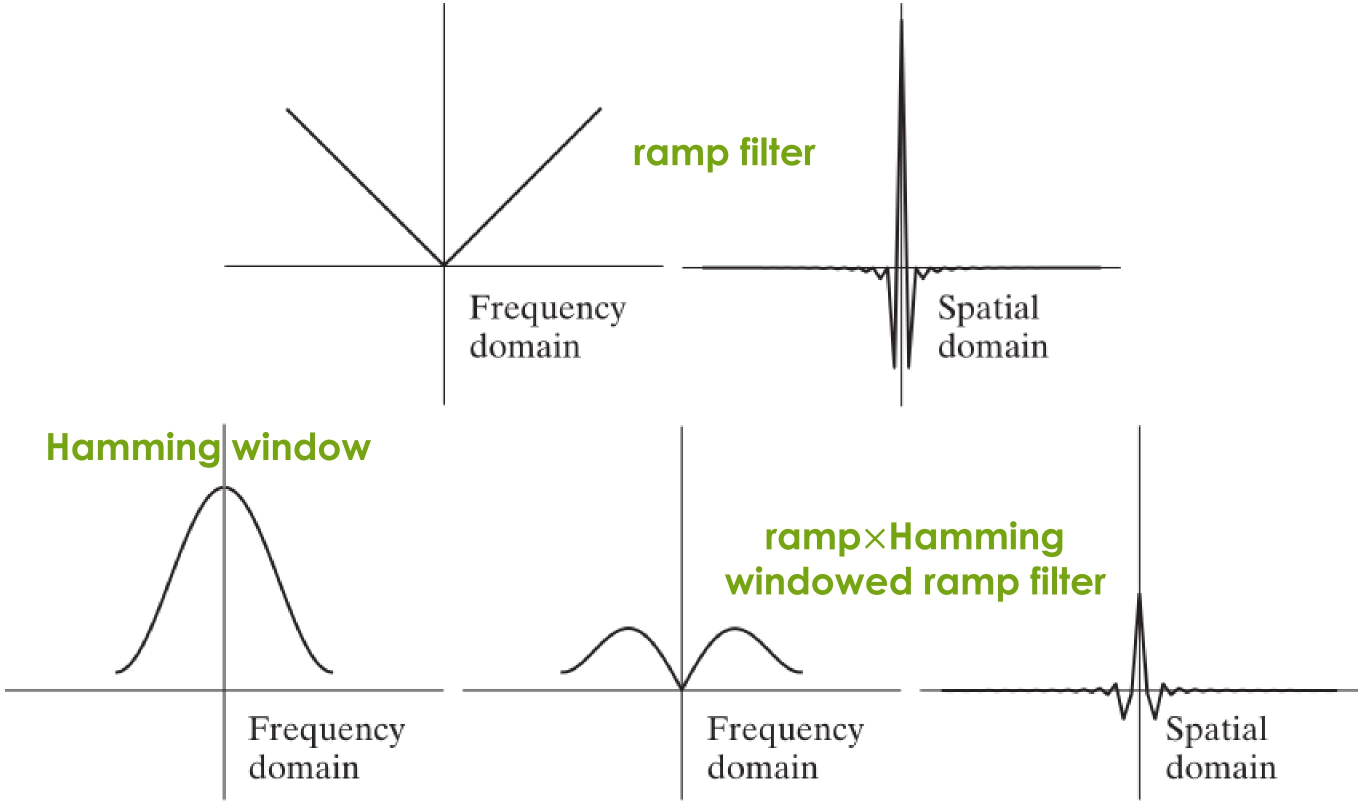
$$f(x,y) = \int_0^{\pi}\left[\int_{-\infty}^{\infty} |\omega| G(\omega,\theta)\, e^{i2\pi\omega\boldsymbol{\rho}} d\omega\right]_{\rho = x\cos\boldsymbol{\theta}+\boldsymbol{y}\sin\boldsymbol{\theta}} d\theta$$

Radon & Fourier

[Ramachandran and Lakshminarayanan 1971]

Reconstruction using parallel-beam filtered backprojections

$|\omega|$



$$h(\omega) = \begin{cases} c + (c-1)\cos\dfrac{2\pi\omega}{M-1} & 0 \le \omega \le (M-1) \\ 0 & \text{otherwise} \end{cases} \qquad \begin{matrix} \text{Hamming window } c = 0.54 \\ \text{Hann window } c = 0.5 \end{matrix}$$

Radon & Fourier

Reconstruction using parallel-beam filtered backprojections

$$f(x,y) = \int_0^{\pi} \left[\int_{-\infty}^{\infty} |\omega| G(\omega,\theta)\, e^{i2\pi\omega\boldsymbol{\rho}} d\omega \right]_{\rho = x\cos\boldsymbol{\theta} + \boldsymbol{y}\sin\boldsymbol{\theta}} d\theta$$

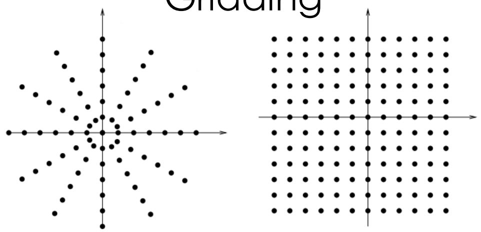
$$\mathcal{F}_{1D}^{-}\{|\omega|\} \equiv s(\rho)$$

$$f(x,y) = \int_0^{\pi} [s(\rho) * g(\rho,\theta)]_{\rho = x\cos\boldsymbol{\theta} + \boldsymbol{y}\sin\boldsymbol{\theta}}\, d\theta$$

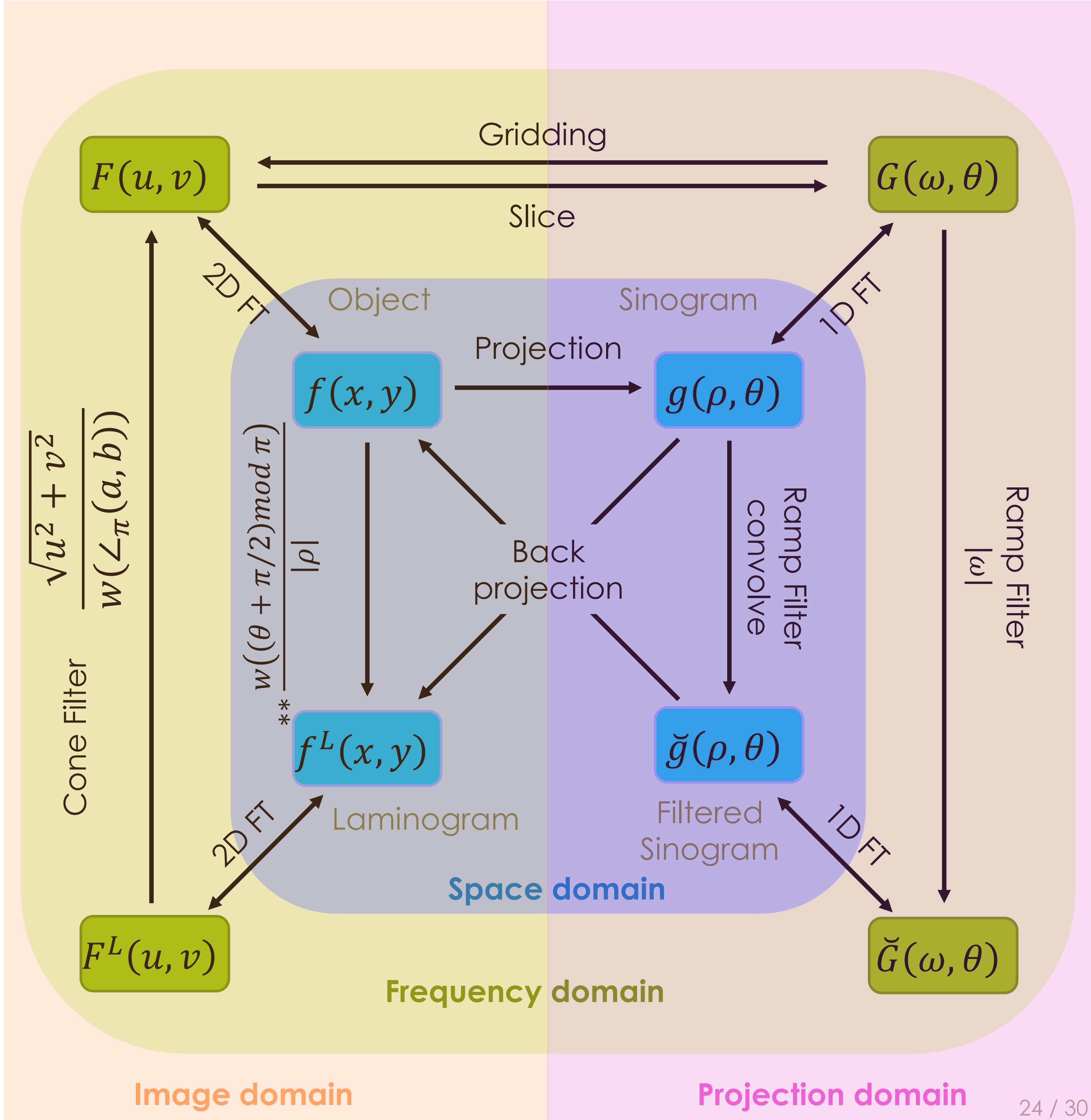
$$f(x,y) = \int_0^{\pi} \left[\int_{-\infty}^{\infty} g(\rho,\theta) s(x\cos\theta + y\sin\theta - \rho)\, d\rho \right] d\theta$$

$$\angle_\pi(a,b) \triangleq \begin{cases} \tan^{-1}\left(\frac{b}{a}\right) & ab > 0 \\ 0 & b = 0 \\ {}^{\pi}/_{2} & a = 0, b \neq 0 \\ \tan^{-1}\left(\frac{b}{a}\right) + \pi & ab < 0 \end{cases}$$

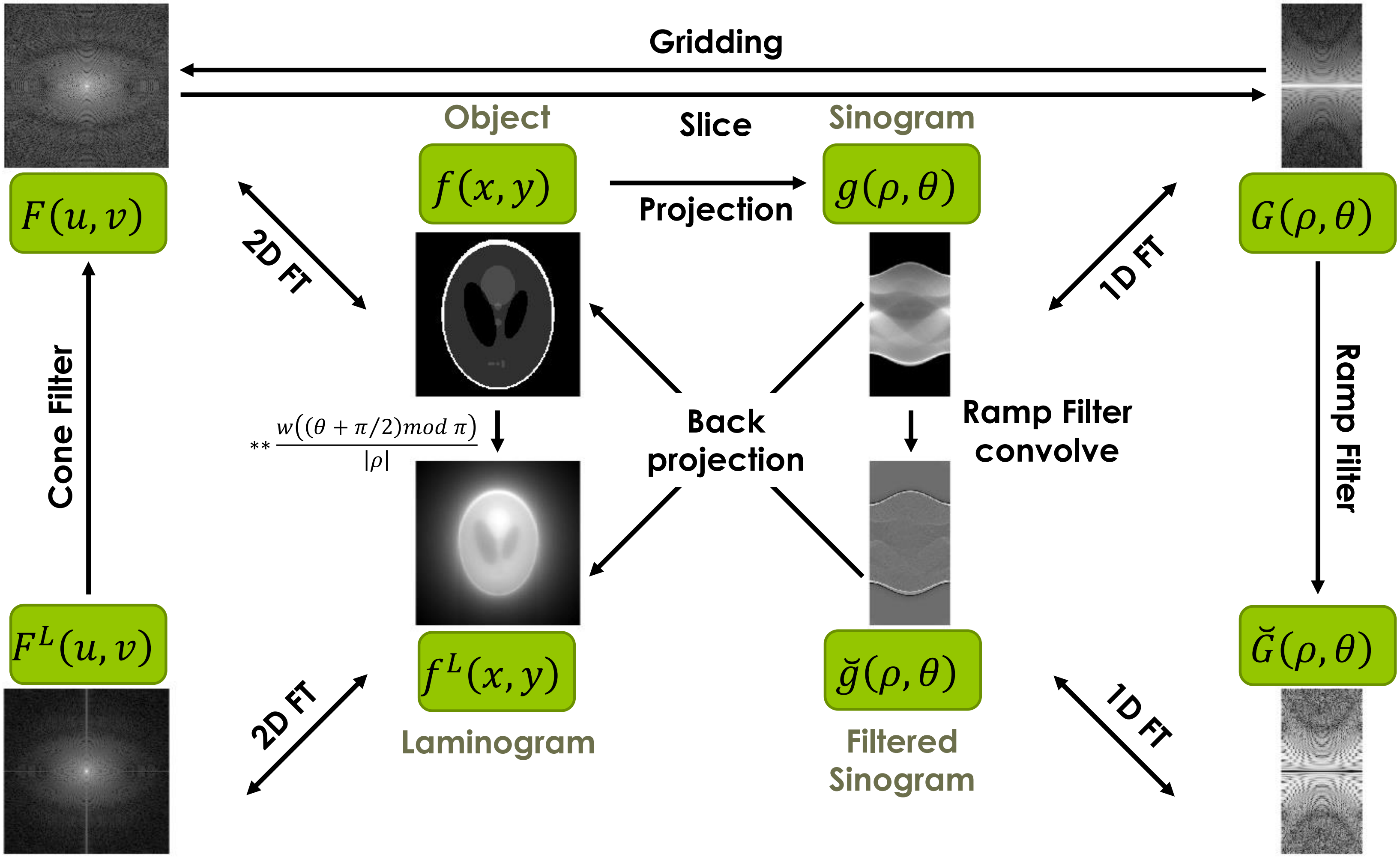
Gridding



$$f(x,y) ** h(\rho) = \iint f(x-s, y-t)\, h\left(\sqrt{s^2+t^2}\right) ds\, dt$$

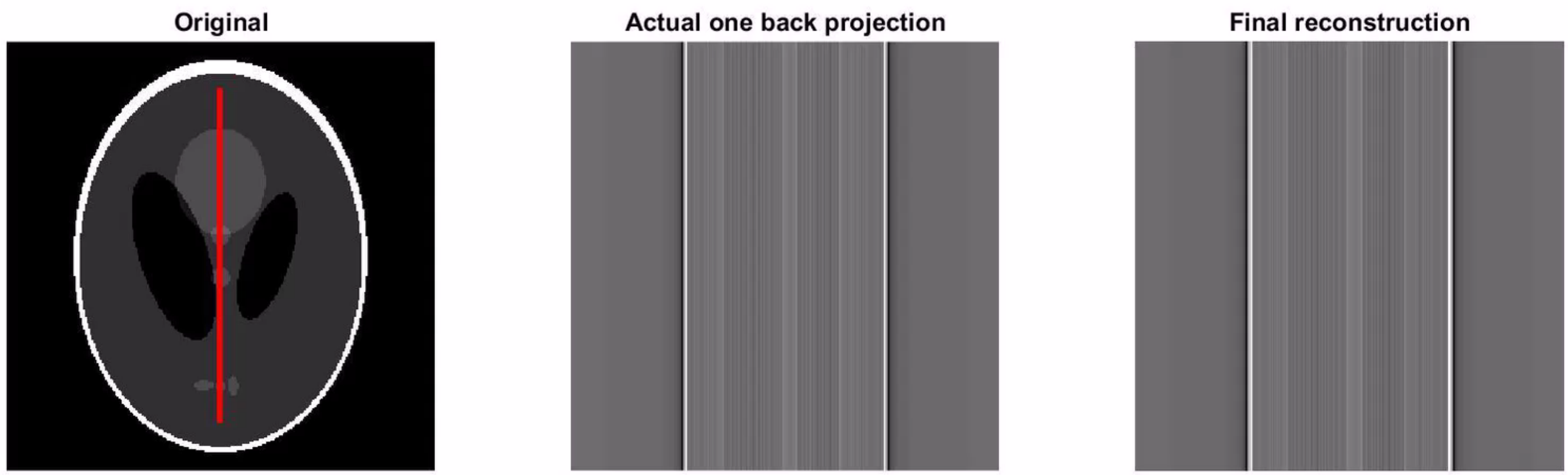


Relationships between projections and transforms

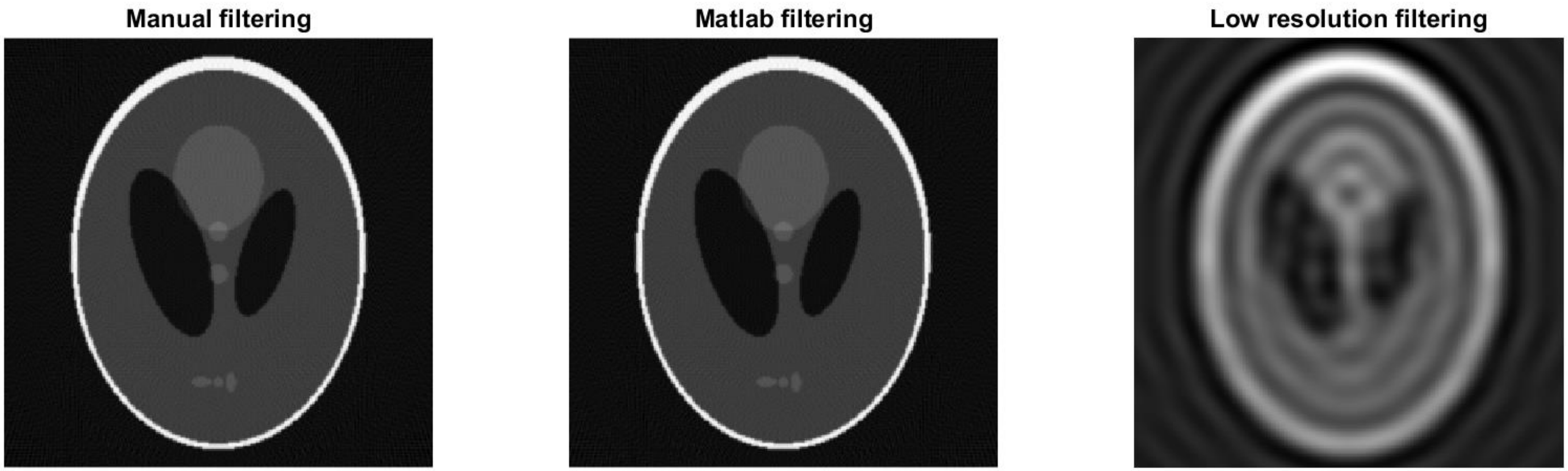


Computed Tomography

Filtered back projection

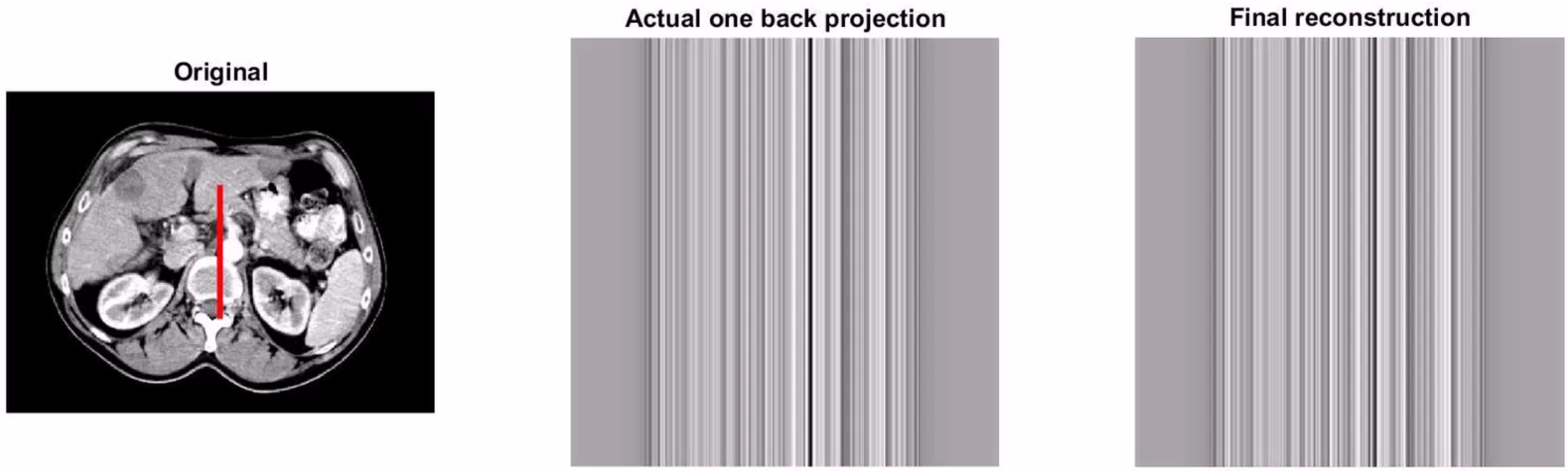


Computed Tomography

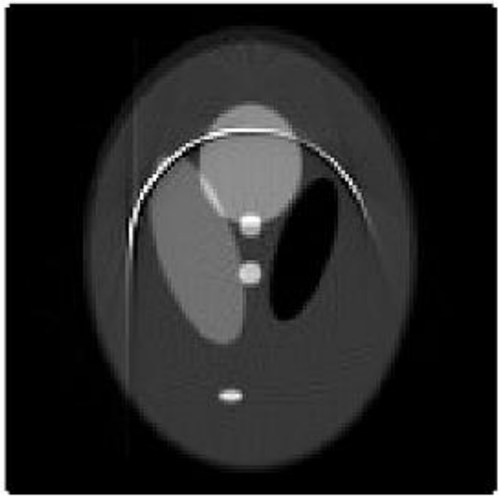


Hough & Radon & Fourier

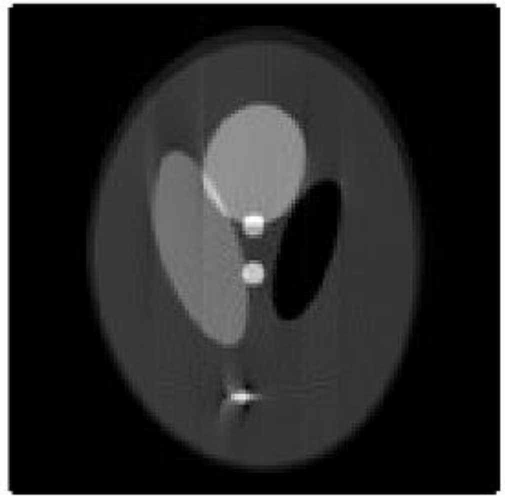
Filtered back projection



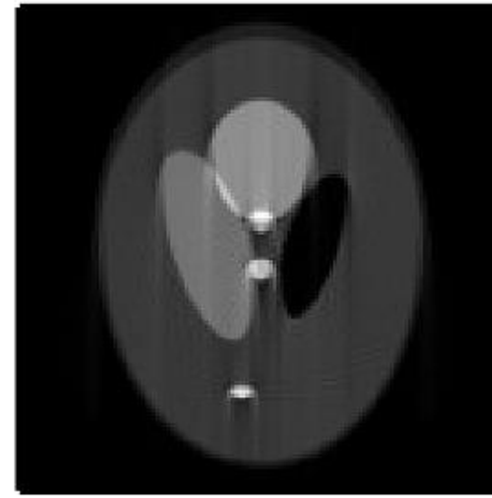
angular range less than π

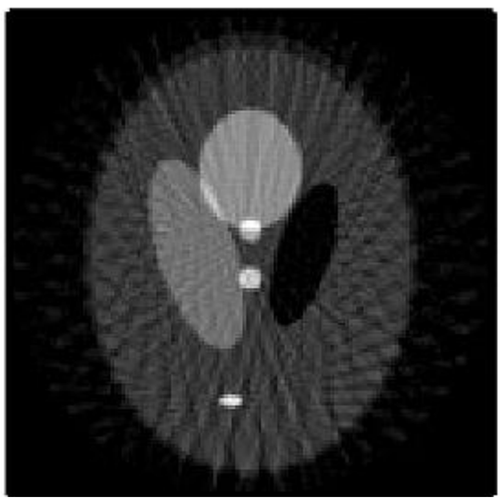


under-sampled angles

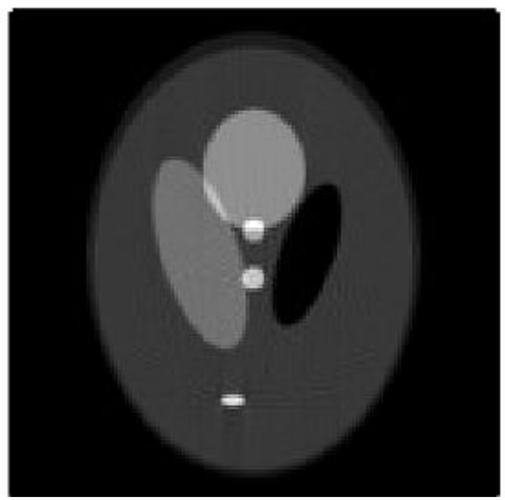


miscalibrated gain of detector

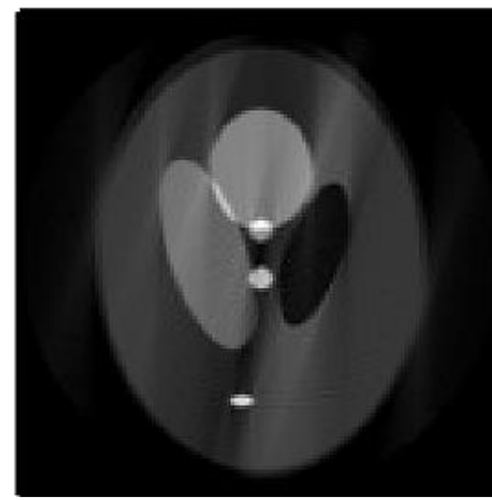




Misaligned detector radialshift



Projection views over $[0, \pi]$



Fan-beam data into parallel-beam reconstructor

Thank you for attention!

Friday seminars
ZOI – UTIA ~ 8 January 2019

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