Discretization of Optimization Problems

Optimization x E-L equation

- Optimization problem

$$\min F(u) = \min \int f(x, u, \nabla u) dx$$

- and corresponding E-L equation

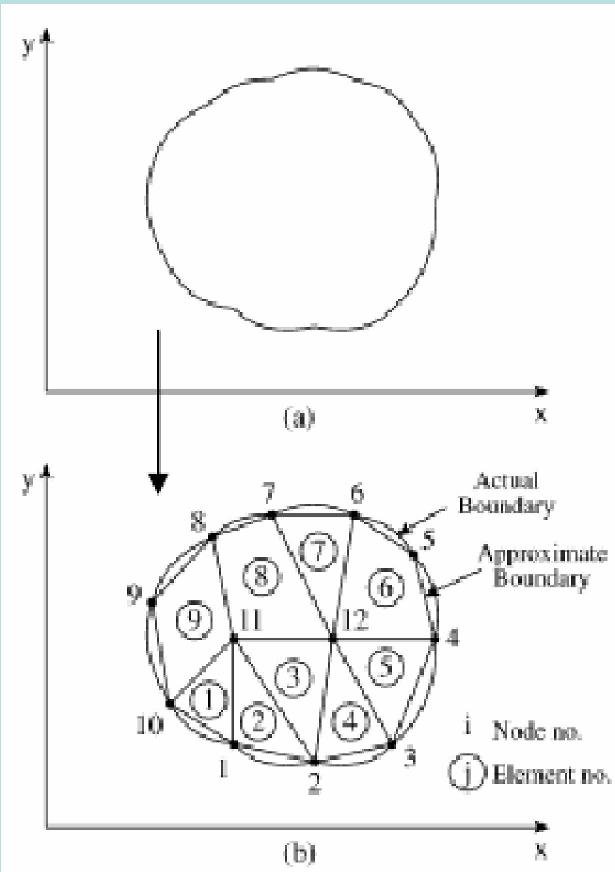
$$F'(u) = 0$$

Discretization

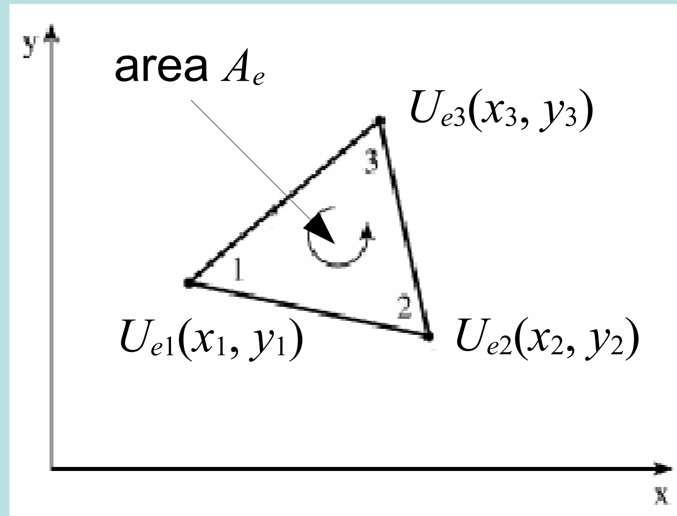
- Finite Element Method (FEM)
  - Solution to integral equation
- Finite Difference Method (FDM)
  - Solution to E-L equation

Finite Element (FEM)

Partition of the space



Triangular Finite Element



$$u \simeq \sum_{e=1}^{N} u_e$$

$$u_e(x, y) = a + bx + cy$$

$$\begin{bmatrix} U_{e1} \\ U_{e2} \\ U_{e3} \end{bmatrix} = \begin{bmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

FEM

- Linear approximation of the function inside each element using values in nodes:

$$u_e(x,y) = \begin{bmatrix} 1 & x & y \end{bmatrix} \frac{1}{2|A_e|} \begin{bmatrix} (x_2y_3 - x_3y_2) & (x_3y_1 - x_1y_3) & (x_1y_2 - x_2y_1) \\ (y_2 - y_3) & (y_3 - y_1) & (y_1 - y_2) \\ (x_3 - x_2) & (x_1 - x_3) & (x_2 - x_1) \end{bmatrix} \begin{bmatrix} U_{e1} \\ U_{e2} \\ U_{e3} \end{bmatrix}$$

or

$$u_e = \sum_{i=1}^{3} \alpha_i(x,y) U_{ei}$$

- Function u is approximated in the whole element not just in the nodes.

Solution of Laplace's equation

- Functional: $F = \frac{1}{2}\int_\Omega |\nabla u|^2 dx$

$$f(x, u, \nabla u) = |\nabla u|^2 = u_x^2 + u_y^2$$

- E-L equation: $-\Delta u = 0$ $\quad\quad \Delta u = u_{xx} + u_{yy}$

- FEM: $F = \sum_e F_e = \sum_e \frac{1}{2}\int_{A_e} |\nabla u_e|^2 dx$

$$u_e = \sum_{i=1}^{3} \alpha_i(x, y) U_{ei} \qquad \nabla u_e = \sum_{i=1}^{3} \nabla \alpha_i(x, y) U_{ei}$$

- $F = \sum_e F_e = \sum_e \frac{1}{2}\int_{A_e} |\nabla u_e|^2 dx$

$$u_e = \sum_{i=1}^{3} \alpha_i(x,y)U_{ei}$$

$$\nabla u_e = \sum_{i=1}^{3} \nabla\alpha_i(x,y)U_{ei}$$

$$F_e = \frac{1}{2}\sum_{i=1}^{3}\sum_{j=1}^{3} U_{ei}\left(\int_{A_e}\langle\nabla\alpha_i,\nabla\alpha_j\rangle dx\right)U_{ej}$$

$$\mathbf{u}_e = \begin{bmatrix} U_{e1} \\ U_{e2} \\ U_{e3} \end{bmatrix} \quad \mathbf{C}^{(e)} = \begin{bmatrix} c_{11}^e & c_{12}^e & c_{13}^e \\ c_{21}^e & c_{22}^e & c_{23}^e \\ c_{31}^e & c_{32}^e & c_{33}^e \end{bmatrix}$$

$$F_e = \frac{1}{2}\mathbf{u}_e^T \mathbf{C}^{(e)} \mathbf{u}_e$$

$$c_{ij}^e = \int_{A_e}\langle\nabla\alpha_i,\nabla\alpha_j\rangle dx$$

- Leads to a quadratic form:

$$F = \sum_{e=1}^{N} F_e = \frac{1}{2}\mathbf{u}^T \mathbf{C}\mathbf{u}$$

- $\mathbf{u}$ … vector of all nodes U_{ei}

  $\mathbf{C}$ … matrix of all coefficients $\mathbf{C}^{(e)}$

- Solution:

$$\forall i \; \frac{\partial F}{\partial u_i} = 0 \quad \Rightarrow \quad \mathbf{C}\mathbf{u} = 0$$

A set of linear equations!!! (Dirichlet B.C.?)

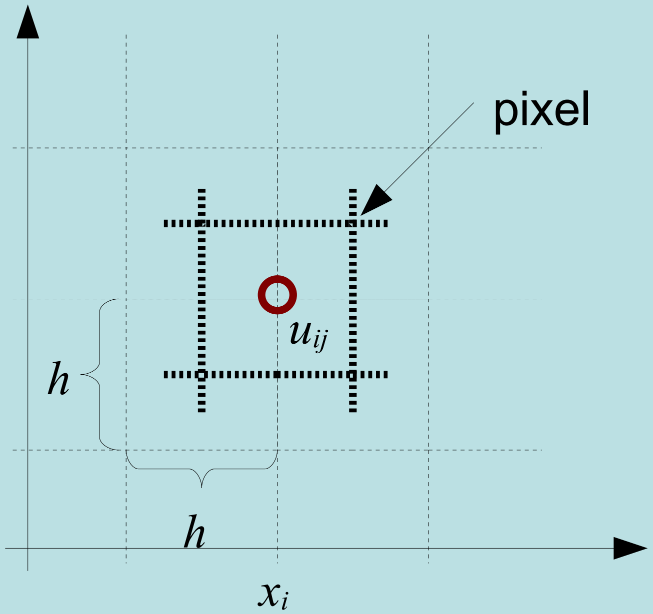
FEM

- Pros
  - Geometrically complex problems
  - Structural mechanics
- Cons
  - Often hard to implement

Finite Difference (FD)

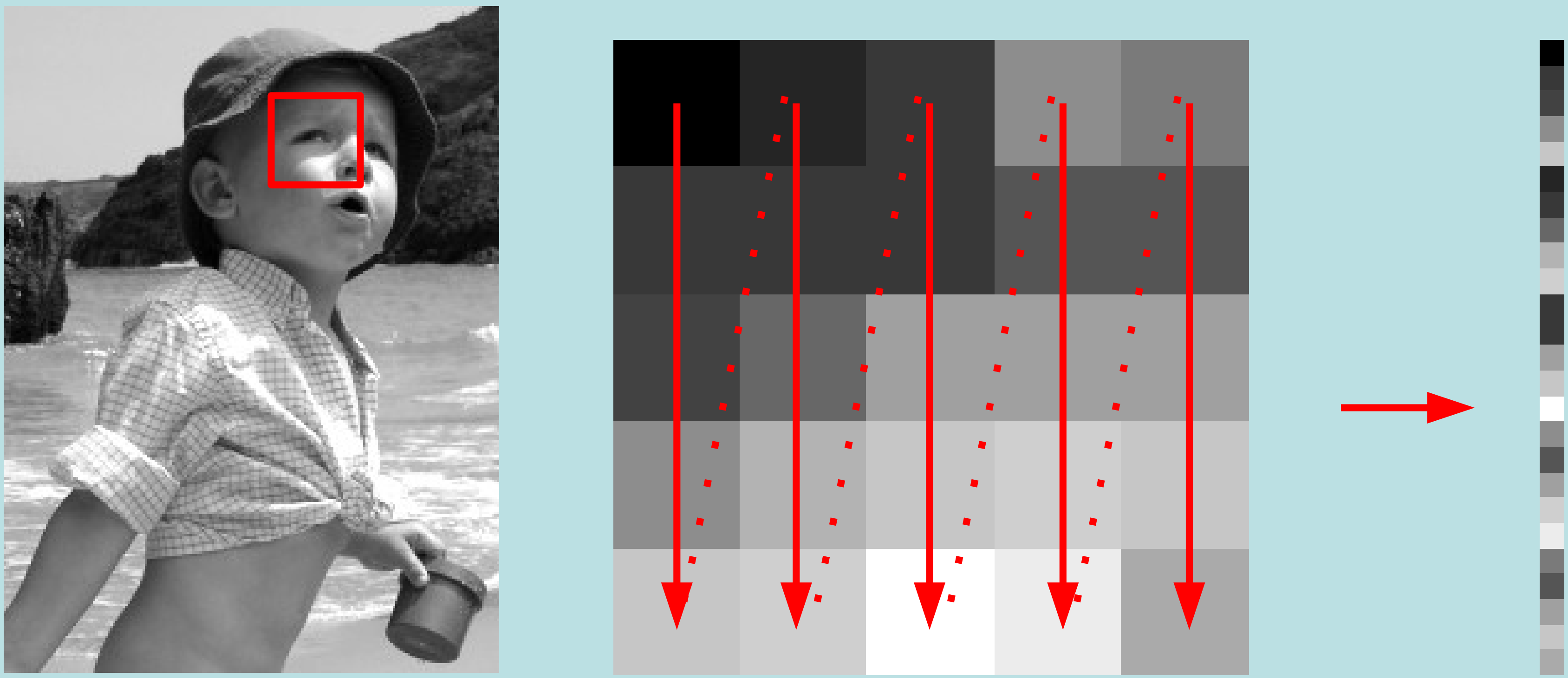
- Nodes on a uniform grid

$$h = 1$$



- Approximation of derivative: $\dfrac{\partial u}{\partial x}, \quad \dfrac{\partial u}{\partial y}$

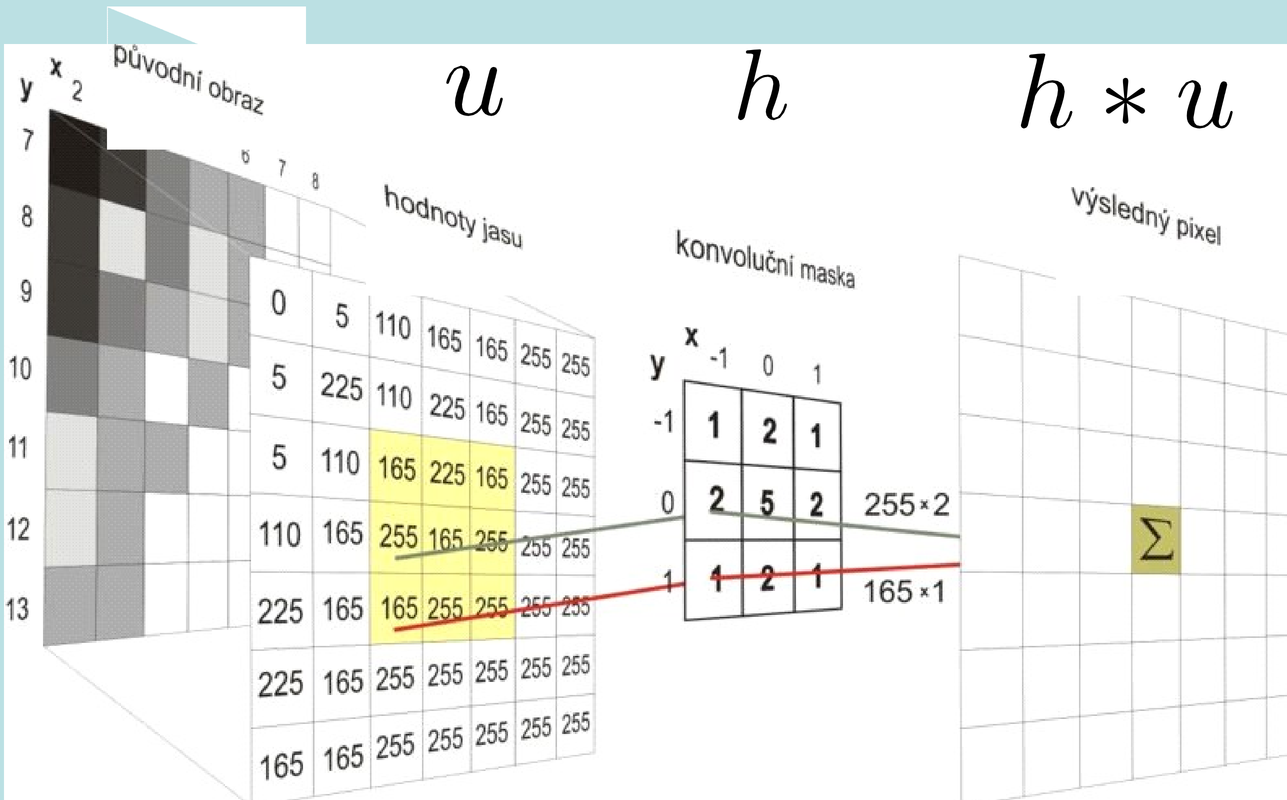
Digital Image



$u(x, y)$ $u_{i,j}$ $\mathbf{u}$

2D convolution

$$h * u = \int u(x-s, y-t)h(s,t)\,ds\,dt$$



Convolution

- Continuous case

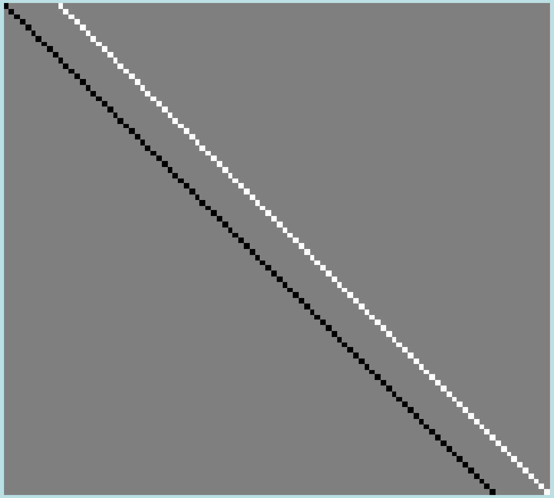
$$h * u = \int h(x - s, y - t)u(s, t)dsdt$$

- Discrete case

$$h * u \approx \mathbf{Hu}$$

$$\mathbf{h} = [1, -1]$$

$$\mathbf{H} =$$



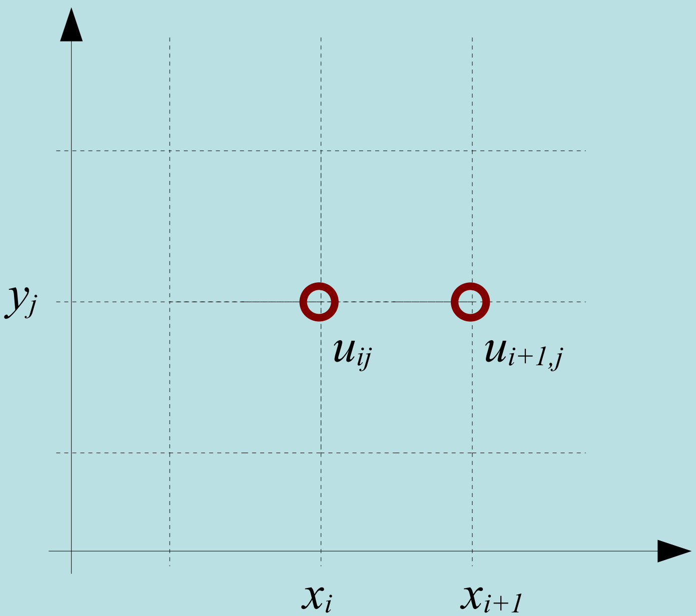
- Boundary effect!!!

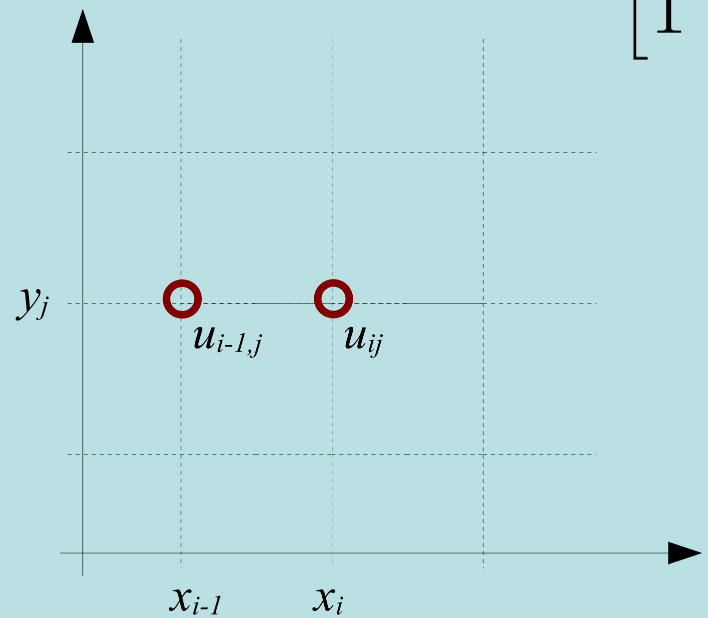
Approximation of Derivatives

- Based on the Taylor expansion

conv. kernel:

$$\begin{bmatrix} 1 & -1 \end{bmatrix}$$





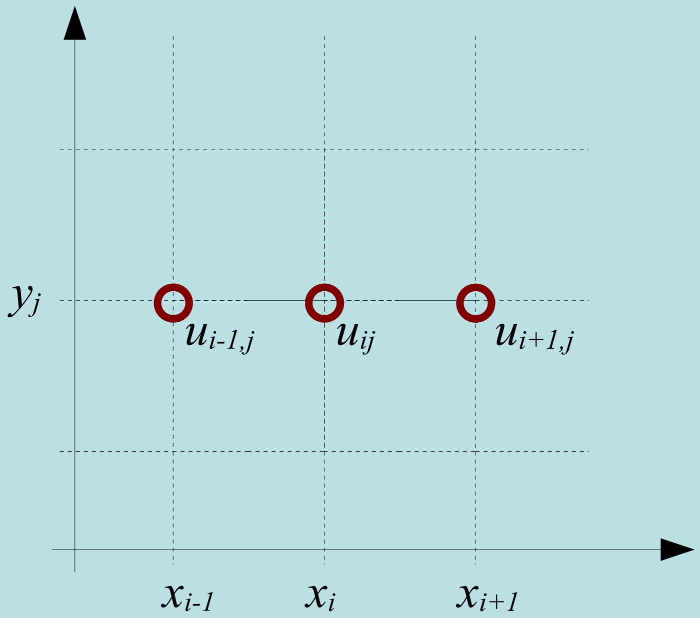
forward difference

$$\frac{\partial u}{\partial x}(x_i, y_i) \approx \frac{u_{i+1,j} - u_{ij}}{h}$$

backward difference

$$\frac{\partial u}{\partial x}(x_i, y_i) \approx \frac{u_{i,j} - u_{i-1,j}}{h}$$

Approximation of Derivatives



conv. kernel:

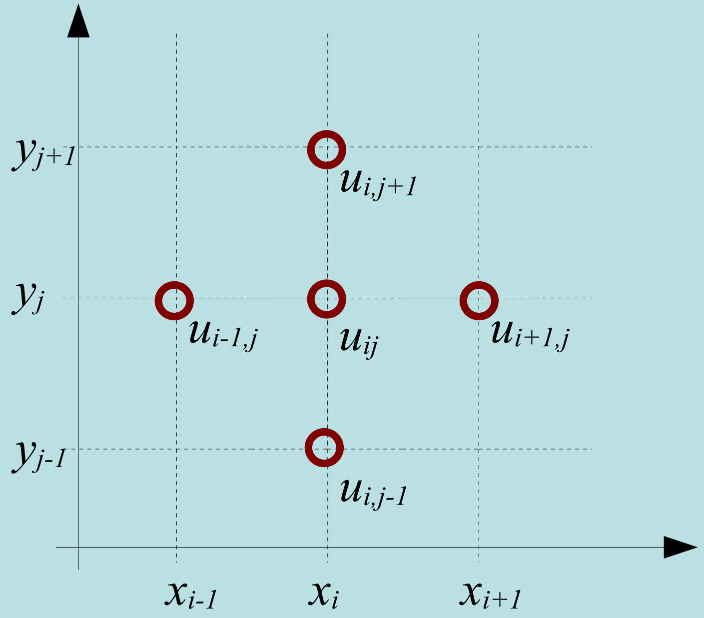
$$\begin{bmatrix} 1 & 0 & -1 \end{bmatrix}$$

centered difference

$$\frac{\partial u}{\partial x}(x_i, y_i) \approx \frac{u_{i+1,j} - u_{i-1,j}}{2h}$$

Approximation of Laplacian

- Apply forward (backward) differences twice both on x and y.



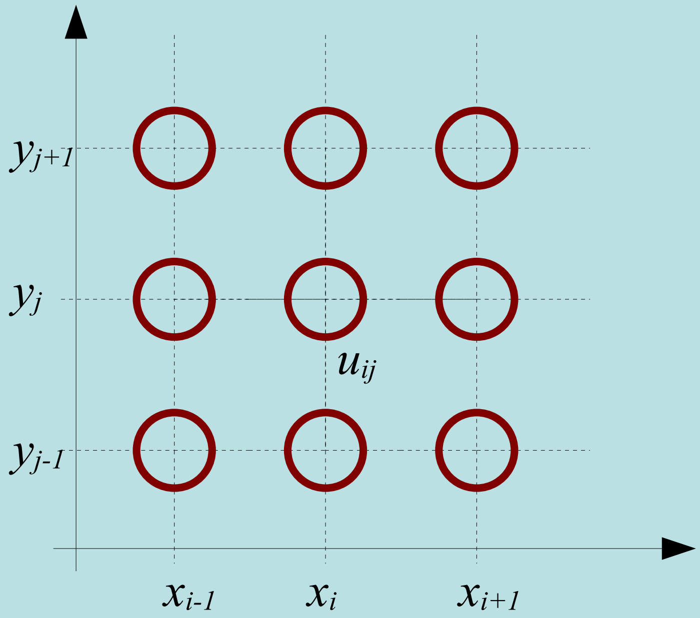
conv. kernel:

$$\begin{bmatrix} 0 & 1 & 0 \\ 1 & -4 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

$$\Delta u(x_i, y_i) \approx \frac{u_{i+1,j} + u_{i-1,j} + u_{i,j+1} + u_{i,j-1} - 4u_{i,j}}{h^2}$$

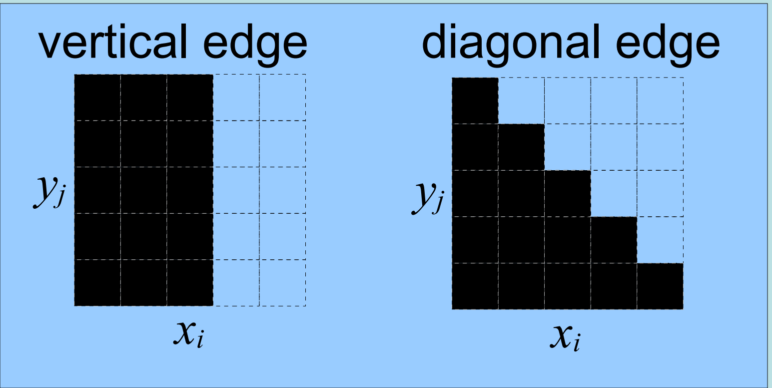
Approximation of Laplacian

- Rotationally invariant



conv. kernel:

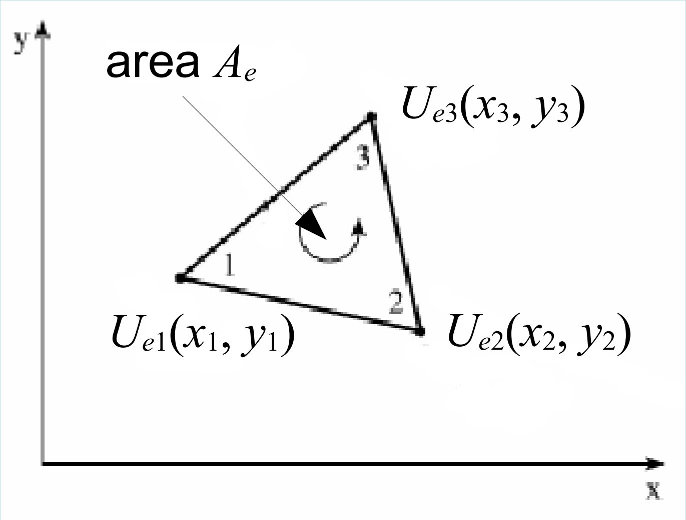
$$\frac{1}{3}\begin{bmatrix} \frac{1}{2} & 2 & \frac{1}{2} \\ 2 & -10 & 2 \\ \frac{1}{2} & 2 & \frac{1}{2} \end{bmatrix}$$



Back to Laplace's equation

- FEM:

  complex partitioning

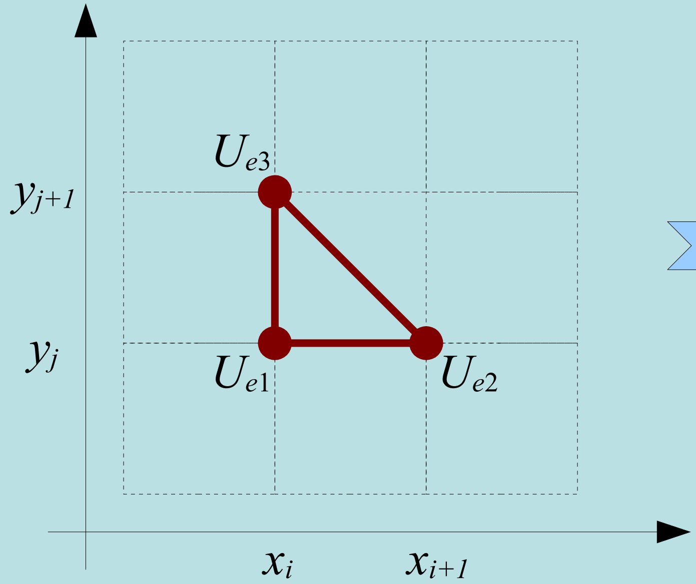


$$F = \sum_{e=1}^{N} F_e = \mathbf{u}^T \mathbf{C} \mathbf{u}$$

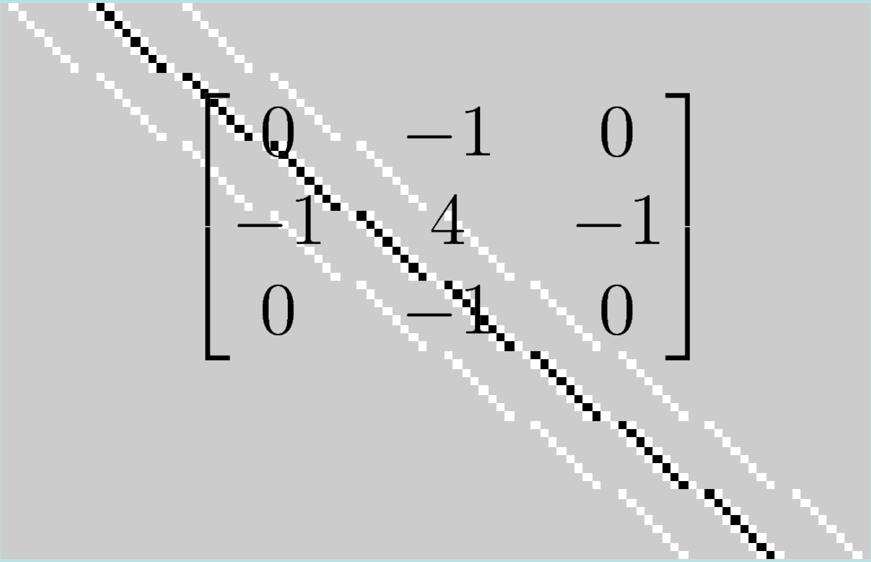
$$c_{ij}^e = \int_{A_e} \langle \nabla \alpha_i, \nabla \alpha_j \rangle dx$$

…

- Regular grid & lin.approx. of u on A



C is a discrete - Laplacian. convolution with



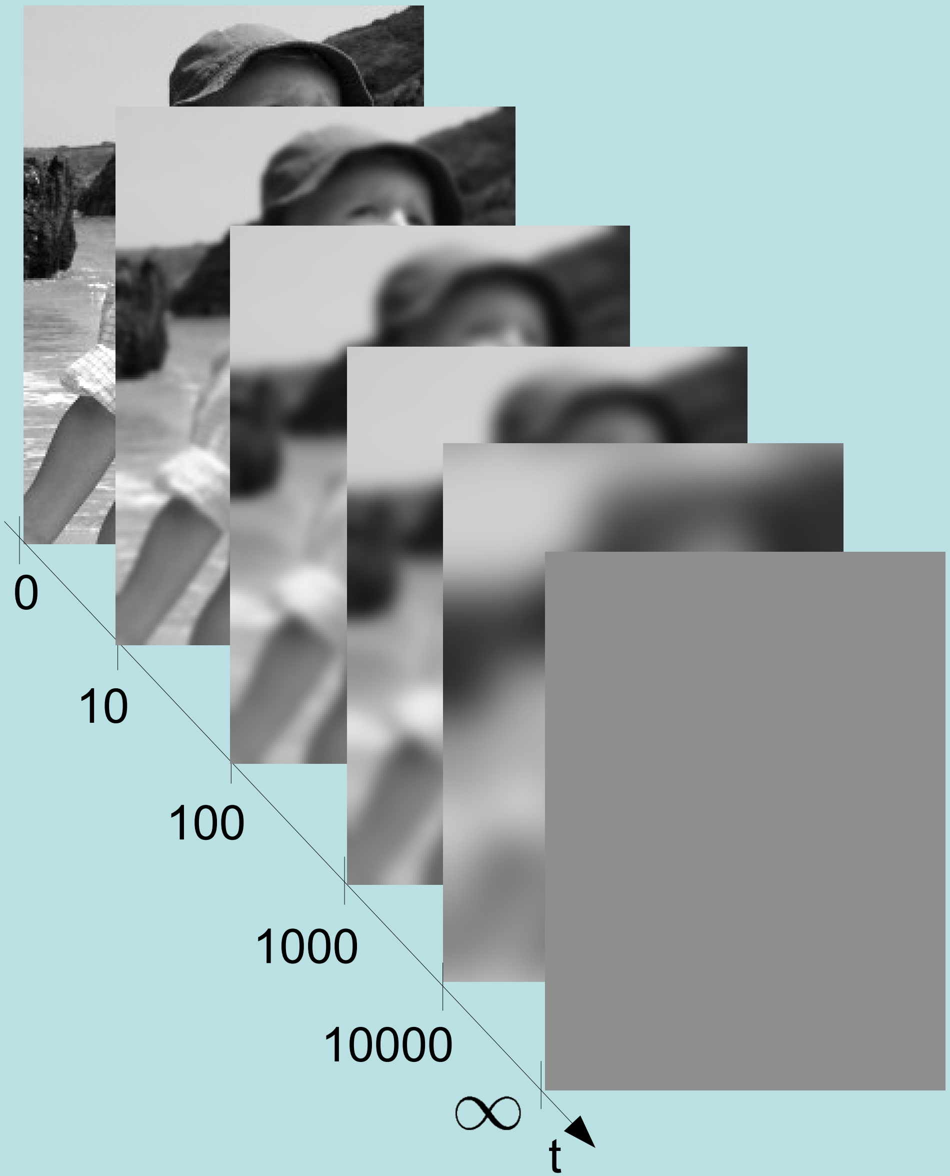
$$\begin{bmatrix} 0 & -1 & 0 \\ -1 & 4 & -1 \\ 0 & -1 & 0 \end{bmatrix}$$

$$
\begin{array}{ccc}
\min \frac{1}{2} \int_\Omega |\nabla u|^2 dx & \xrightarrow{\text{FEM}} & \min \frac{1}{2} \mathbf{u}^T \mathbf{C} \mathbf{u} \\
\text{E-L eq.} \Big\downarrow & & \Big\downarrow \nabla_{\mathbf{u}} \\
-\Delta u = 0 & \xrightarrow{\text{FD}} & \mathbf{C}\mathbf{u} = 0
\end{array}
$$

Evolution of Laplace's Equation

$$u_t = \Delta u$$

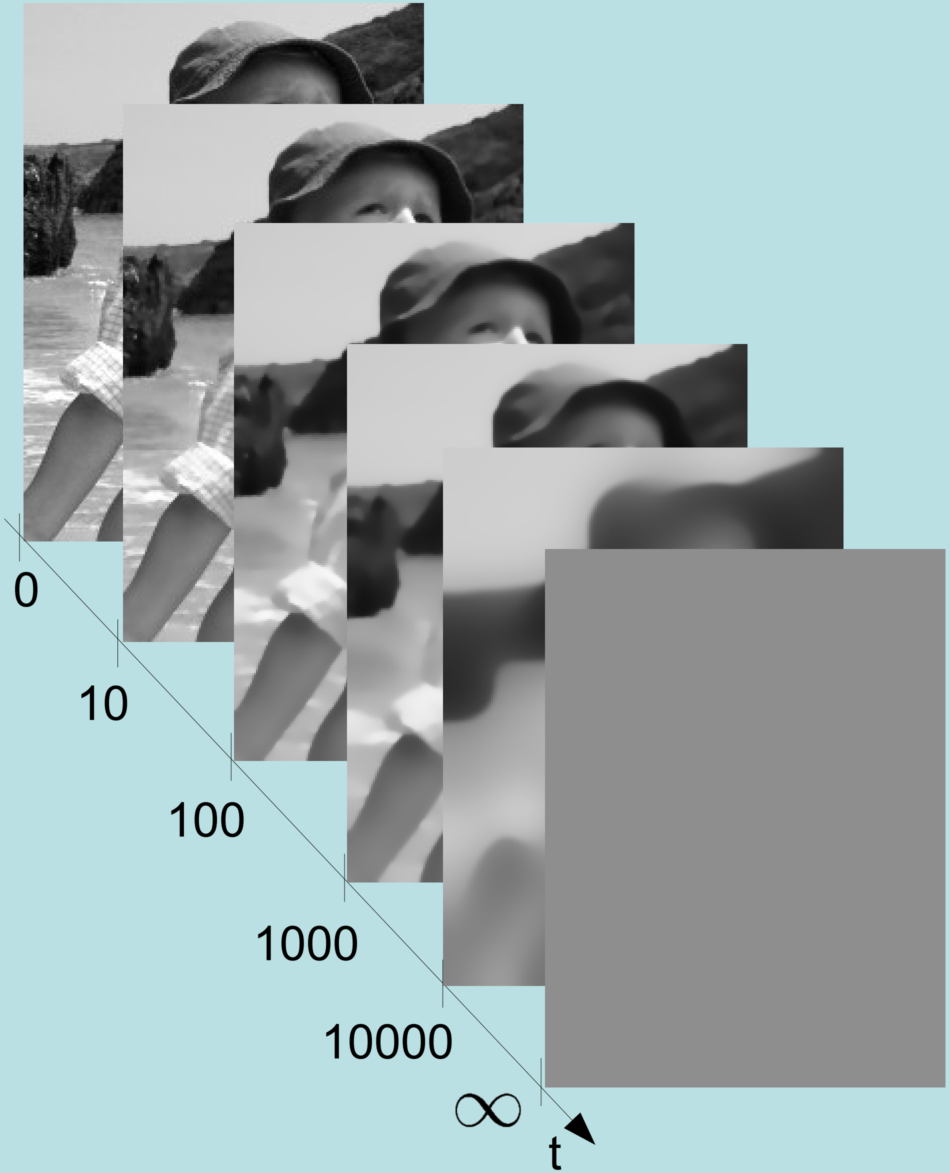
$$\mathbf{u}_{k+1} = \mathbf{u}_k - \alpha \mathbf{C} \mathbf{u}_k$$



Evolution of TV Equation

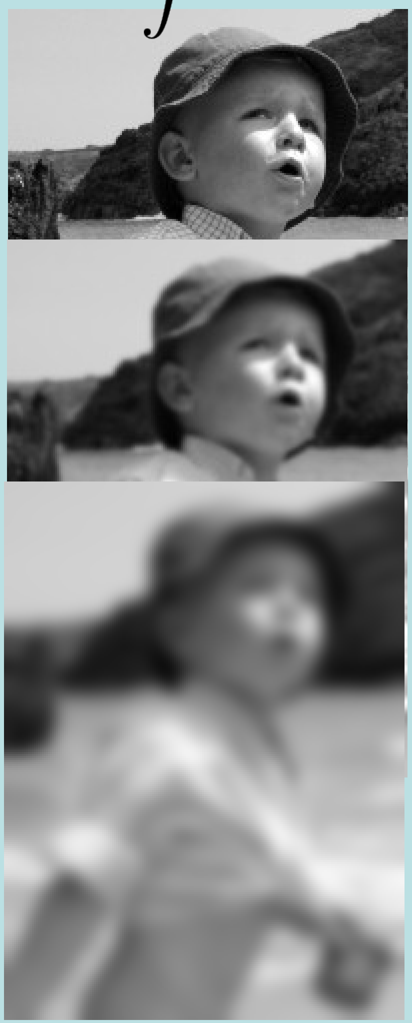
$$u_t = \mathrm{div}\Big(\frac{\nabla u}{|\nabla u|}\Big)$$

$$\mathbf{u}_{k+1} = \mathbf{u}_k - \alpha \mathbf{L}_{\nabla \mathbf{u}_k} \mathbf{u}_k$$



Isotropic & Anisotropic Diffusion

$$\min \int |\nabla u|^2$$



$$\min \int |\nabla u|$$